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JEL Classification: G11, G12, G13.
Keywords: Subjective model uncertainty, signal mean and precision, option-implied uncertainty premium, variance risk premium, trading volume.

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Abstract

This paper develops a dynamic model of prices and trades in a risky security and an option, where agents use different subjective likelihood functions to interpret a public signal, but they are initially uncertain about the signal's mean or precision. Such a framework of subjective model uncertainty can explain the seemingly overpriced options and endogenously generate variance risk premium (VRP). However, the model yield contrasting implications on whether options and VRP are spanned and on whether trading volume is positively related to VRP. Empirical evidence largely supports subjective model uncertainty about the signal precision in major futures markets.

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1 Introduction

Uncertainty is one of the most salient features of financial markets. Variance risk premium (VRP)—the difference between option-implied and expected variances—is usually interpreted as the compensation for economic uncertainty (Bollerslev, Tauchen, and Zhou, 2009; Drechsler, 2012; Bekaert and Hoerova, 2016; Zhou, 2017). Recently, there are some puzzling findings that index options seem to be overpriced after adjustment for stock risk factors (Constantinides, Czerwonko, Jackwerth, and Perrakis, 2011), and that, more generally, VRP and options prices are not completely spanned by underlying security prices (Collin-Dufresne and Goldstein, 2002; Carr and Wu, 2009). Endowment uncertainty (as in Bollerslev, Tauchen, and Zhou, 2009) and information uncertainty (as in Buraschi and Jiltsov, 2006) are unlikely to explain these phenomena, because they shall affect the risk premium in the underlying security's price at the same time. The purpose of this paper is to examine whether subjective model uncertainty is able to explain these phenomena, by analyzing its effects on the prices and trades in a risky security and an option.

In this paper, subjective model uncertainty refers to the situation in which agents use their subjective models (or likelihood functions) to interpret a public signal, but they are initially uncertain (or indecisive) about some model parameters (e.g., signal mean and precision). This concept is motivated by psychology research findings that ambivalent people have simultaneous conflicting reactions, beliefs, or feelings towards some object, such that they face uncertainty or difficulty to make decisions. However, ambivalent attitudes are susceptible to transient information (e.g., mood), which results in a more malleable evaluation (Kaplan, 1972; Bell and Esses, 1997). In reality, because of ambivalent attitudes, many consumers find it difficult to make purchase decisions (Otnes, Lowrey, and Shrum, 1997).

We analyze a three-period model with subjective model uncertainty. A risk-free bond, a risky security, and a quadratic option on this security are available for trading among competitive risk-averse agents. In Period 3, the security's liquidation value is realized, and the option yields a payoff of the square of the security's liquidation value. In Period 2, the agents observe a public signal and interpret the signal with their own subjective models (or likelihood functions). In Period 1, agents are homogeneous with zero endowments in the two risky assets and are uncertain (or indecisive) about some parameters of their subjective models (either signal mean or signal precision).

There are two layers of uncertainty in Period 1. First, agents are uncertain about the model parameters of the average agent, who is defined as holding the average parameters. Second, agents are uncertain about their own types, i.e., they are uncertain about the relative values of their own model parameters in comparison with those of the average agent. These assumptions imply that, in Period 2, agents disagree on signal mean (i.e., differ in their optimism levels) or disagree on signal precision (i.e., differ in their confidence levels), and they know their own types.[1]

In Period 2, the security and option prices are determined as if by the average agent who trades neither of these two risky assets. Agents trade because of their different opinions about the signal. In Period 1, the agents do not trade. Subjective model uncertainty about either signal mean or signal precision leads to a genuine negative uncertainty premium embedded in option price (or a more expensive option) and a positive variance risk premium (VRP), contrary to the case where agents know their subjective models in Period 1. Subjective model uncertainty is therefore an endogenous approach to generate VRP, while the stochastic volatility model, e.g., Heston (1993), is an exogenous approach to generate VRP.

Intuitively, an agent perceives that as long as he deviates from the average agent in Period 2, he will profit from the "mispricing" of the option price, which is determined by the average agent. Since agents are competitive, they would have to pay more to purchase the option in Period 1, in the presence of uncertainty about their subjective models. This result implies that the option-implied uncertainty premium origins from agents' uncertainty about their own types (the optimism or confidence levels) rather than from the uncertainty about the average agent's model parameters. Consequently, variance risk premium is not induced by—in fact, is independent of—agents' risk aversion. In other words, variance risk premium

[1]There is no doubt that agents' subjective model parameters are not restricted only to signal mean and signal precision. We focus on these two parameters because they are the most widely studied ones in the literature of difference-of-opinions (see, e.g., Kandel and Pearson, 1995; Cao and Ou-Yang, 2009).

is not "risk premium"per se, rather, it is a pure "uncertainty premium".[2]

However, subjective model uncertainty about signal mean or precision also produce contrasting equilibrium properties. First, the former generates uncertainty premia in both security and option prices, while the latter only generates a uncertainty premium in option price. Second, the former implies that option price, the uncertainty premium, and variance risk premium are spanned completely by the security price, while the latter predicts the opposite. Third, the former predicts that trades occur only in the security, while the latter predicts that trades occur in both the security and option. Forth, the former predicts that trades in the security occur without price change and is always positively related to lagged variance risk premium, while the latter predicts that trading volume in the security is positively related to current-period absolute price change (or return volatility) and is negatively related to lagged variance risk premium. Our model indicates that the puzzling phenomena about option price and variance risk premium may be generated by the subjective model uncertainty about signal precision.

We empirically test the model's novel implications about the relationships between trading volume and variance risk premium, as well as absolute return and implied volatility. The empirical exercise relies on nine major futures markets, in which information asymmetry is less of a concern. Our main findings are that there are stable and consistent negative relationship between the trading volume and lagged variance risk premium or volatility risk premium in three major government bond futures markets (US, Germany, and Japan) and three currency futures markets (Australian Dollar, Euro, and Japanese Yen), which is consistent with the implications from the subjective model uncertainty about signal precision. For three major stock index futures markets (S&P500, DAX, and Nikkei), the results are mixed—lagged variance risk premium positively predicts trading volume, while lagged volatility risk premium negatively predicts trading volume; which suggests a mixture of subjective model

[2]Heston (1993) generates positive volatility (variance) risk premium by assuming a negative correlation between consumption shock and volatility shock. Under CRRA utility, if this correlation is zero, then the volatility (variance) risk premium is zero. Bollerslev, Tauchen, and Zhou (2009) adopt the Epstein-Zin-Weil utility. In their model, the correlation between consumption shock and volatility shock is zero, an early resolution of intertemporal uncertainty of consumption still "economically" generates a positive variance risk premium, which does not vanish even if the risk aversion coefficient equal one.

uncertainty about both signal mean and signal precision.

Our model is closely related to the theoretical literature on difference-of-opinion (see, e.g., Harris and Raviv, 1993; Kandel and Pearson, 1995; Buraschi and Jiltsov, 2006; Cao and Ou-Yang, 2009; Buraschi, Trojani, and Vedolin, 2014, among others). Similar to these models, the agents in our model use different likelihood functions to interpret the public signal in Period 2. However, the focus of our model is to examine the impact of agents' uncertainty about their likelihood functions, and particularly of the uncertainty about agents' types in Period 1. Because of the absence of subjective model uncertainty, those models cannot address the two puzzling facts about variance risk premium and options prices mentioned above. Bessembinder, Chan, and Seguin (1996) empirically analyze the relationship between open interest, a proxy for the dispersion of beliefs, and trading volume. Our paper empirically analyzes the relationship between variance risk premium, a proxy for variation in the dispersion of beliefs, and trading volume.

There is a large literature on the relationship between trading volume and asset volatility. On the one hand, Wang (1994), Llorente, Michaely, Saar, and Wang (2002), among others, offer theoretical models based on information asymmetry. On the other hand, Copeland (1976), Tauchen and Pitts (1983), Gallant, Rossi, and Tauchen (1992), Gerety and Mulherin (1992), Bessembinder and Seguin (1993), Foster (1995), Andersen (1996), Liu, Tauchen, and Zhang (1996), etc., conduct empirical examinations of the volatility-volume relationship. However, neither the theoretical nor the empirical strand of research has focused on the option-implied volatility, let alone variance risk premium.

The rest of the paper is organized as follow. Section 2 analyzes two specific models on subjective model uncertainty: one about signal mean and the other about signal precision. We characterize the equilibrium by solving the agents' optimization problems with backwardation, compare the equilibrium properties of these two models on asset prices, implied volatility, variance risk premium, and trading volume, and derive testable empirical hypotheses. Section 3 empirically tests these hypotheses with nine futures on stock indices, bonds, and currencies. Section 4 concludes. Technical proofs of theoretical results are provided in appendices.

2 The Model

In this section, we present a formal model on subjective model uncertainty in which trades occur because agents use different subjective models (or likelihood functions) to interpret a pubic signal (e.g., a public announcement), but agents are initially uncertain about their subjective models. We assume that every agent holds on to their interpretations, even though other agents have different ones, after agents know their own types. Throughout the paper, we maintain the assumption that agents optimize within the framework of their beliefs.

2.1 Common Model Setups

In this economy, there exist three time periods indexed by $t = \{1, 2, 3\}$. One risk-free bond, one risky security, and an option on this security, are available for trading. Without loss of generality, we normalize the risk-free interest rate to be zero. The economy is composed of a continuum of competitive agents indexed by $i \in [0, 1]$. Let P_t and P_t^Q denote the prices of the security and the option at time t, respectively. Agent i's optimal demand for these two assets at time t are denoted by D_t^i and D_{tQ}^i, respectively.

In Period 3, the risk security's liquidation value μ is realized, and the agents consume their wealth. We assume that μ follows a normal distribution:

$$\mu \sim N(0, 1/K_1), \tag{1}$$

where K_1 is the agents' common prior of the precision of the security's liquidation value. When the security's liquidation value follows a normal distribution, and agents have negative exponential (CARA) utilities, Brennan and Cao (1996) and Cao and Ou-Yang (2009) show that introducing a continuum of call and put options is equivalent to introducing an asset whose payoff is a quadratic function of the liquidation value in completing the market. To generate trading in options in a tractable way, we follow these papers by introducing a single quadratic derivative asset (hereafter the option) with a payoff of $Q(\mu) = \mu^2$ in Period 3.

To emphasize the effect of speculative trading due to different interpretation of public information rather than hedging demand, we assume that each agent has the same initial endowments in the risky assets, which are denoted by x and x_Q, respectively. We further set $x = 0$ and $x_Q = 0$ to indicate that the agents' speculative trading in these assets is a zero-sum game. We can interpret the risky security as a futures contract, as we do later in our empirical tests.

Agents observe a public signal θ in Period 2. They use different models (or likelihood functions) to interpret this public signal, and thus disagree about the relationship between μ and θ. Specifically, the public signal is given by the security's liquidation value plus a noise term and agents disagree about the distribution of the noise term:

$$\theta = \mu + \eta_2^i, \qquad \eta_2^i \sim N(m_2^i, 1/n_2^i), \tag{2}$$

where η_2^i follows a normal distribution with mean m_2^i and variance $1/n_2^i$, and is independent of μ. Here, m_2^i and n_2^i are agent i's subjective model parameters (mean and precision, respectively). While the public signal is observed in Period 2, agents know their likelihood functions (or model parameters) between Period 1 and Period 2. They are homogeneous and uncertain about their subjective models in Period 1, i.e., they only know the distributions of their model parameters m_2^i and n_2^i.

To facilitate the discussion, we define the average agent in the economy, as well as the optimism (pessimism) level and the confidence level of each agent about the public signal as follows:

Definition 1. *The average agent is defined as an agent who holds the average model parameters. That is, his belief regarding the signal's mean equals $m \equiv \int_0^1 m_2^i di$ and his belief regarding the signal precision equals $n \equiv \int_0^1 n_2^i di$. When $m_2^i > m$ ($m_2^i < m$), agent i is optimistic (pessimistic) about the public signal. When $n_2^i > n$ ($n_2^i < n$), agent i has high (low) confidence about the public signal.*

One way to interpret subjective model uncertainty is that the agents have ambivalent attitudes in Period 1, such that it is difficult for them to make decisions about their subjective models. Some external stimulus is realized between Period 1 and Period 2, leading to the agents' decisions of their models. This interpretation is motivated by psychology research findings that ambivalent persons have simultaneous conflicting reactions, beliefs, or feelings towards some object, such that they face uncertainty or difficulty to make decisions; however, ambivalent attitudes are susceptible to transient information (e.g., mood), which can result in a more malleable evaluation.

Agent i has a negative exponential (CARA) utility function, which is defined on his wealth W_3^i in Period 3, $U(W_3^i) = -\exp(-W_3^i/\tau)$, where τ denotes the coefficient of risk tolerance. In each period, the objective of agent i is to choose his positions in the security D_t^i and the option D_{tQ}^i to maximize his expected utility conditional on his information set:

$$\max_{\{D_t^i, D_{tQ}^i\}} E_t^i[U(W_3^i)], \tag{3}$$

where $E_t^i[.]$ denotes the expectation algorithm based on agent i's information set in period $t \in \{1, 2\}$. The CARA utility function is widely adopted in the literature on difference-of-opinion, e.g., Harris and Raviv (1993) and Kandel and Pearson (1995), which enables us to obtain closed-form expressions for the prices of the security and option in the presence of subjective model uncertainty.

In general, it is very complicated to tackle agents' heterogeneous likelihood functions. In the following two subsections, we simplify the task by examining two widely studied cases in the literature: First, agents disagree on the signal mean but agree on the signal precision in Period 2 (see, e.g., Kandel and Pearson, 1995); second, agents agree on the signal mean but disagree on the signal precision in Period 2 (see, e.g., Cao and Ou-Yang, 2009). For each case, we specify additional assumptions, characterize the equilibrium, and discuss the corresponding equilibrium properties.

Throughout the paper, we refer superscript "*" to the risk neutral measure, superscript "a" to the average agent, superscript (or subscript) "i" to agent i, and subscript (or superscript) "Q" to the option. Subscript "t" usually refers to the time, where $t \in \{1, 2, 3\}$. Since agents are homogeneous in Period 1, for simplicity, we use $E[.]$ to denote the expectation algorithm based on the information set of agent i in Period 1.

2.2 Case I: Belief Uncertainty about Signal Mean

In this subsection, we examine the specification of subjective model uncertainty about the mean of a public signal. Agents agree on the signal precision in Period 2 ($n_2^i = n$). Agent

i's belief regarding the signal mean in Period 2 satisfies:

$$m_2^i = m_c + \alpha^i, \tag{4}$$

where $i \in [0,1]$, $m_c \sim N(0, \sigma_m^2)$, and $\alpha^i \sim N(0, \sigma_\alpha^2)$, are random variables independent of each other. Here, m_c and α^i represent the common and idiosyncratic components of agent i's belief about the signal mean, respectively. By calculation using Definition 1, $m = m_c$. Since $\alpha^i = m_2^i - m$, σ_α measures the variation in agents' dispersion of beliefs or the uncertainty of agents' optimism (pessimism) level.

Variance risk premium is usually defined as the difference between option-implied and expected return variance and is interpreted as a compensation for return variance uncertainty. To quantify the variance risk premium, we need to specify the realized distribution of the signal (in this case, the realized mean) and its relationship with agents' subjective beliefs. As shown formally in Theorem 1, the prices of the security and the option are determined by the average agent once all agents observe the signal. Hence, it is sensible to assume that the realized mean of the signal m_2^r equals the subjective belief of the average agent regarding the signal mean, that is,

$$m_2^r = m. \tag{5}$$

where m_2^r follows a normal distribution with mean 0 and variance σ_m^2.

The equilibrium is solved with backwardation. The equilibrium prices and trades are summarized in the following Theorem.

Theorem 1. *In the economy with subjective model uncertainty about the signal mean in Period 1, there exists a unique equilibrium in which prices are given by*

$$P_2 = \mu_2, \quad P_{2Q} = P_2^2 + \frac{1}{K_2}, \tag{6}$$

$$P_1 = \mu_1 + \omega_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1} + \omega_{1Q}, \tag{7}$$

the demands of agent i for the security and option are given by

$$D_{2Q}^i = 0, \quad D_2^i = \tau(\mu_2^i - \mu_2)K_2, \tag{8}$$

$$D_{1Q}^i = 0, \quad D_1^i = 0, \tag{9}$$

and the uncertainty premia for the security and option in Period 1, $-\omega_1$ and $-\omega_{1Q}$, satisfy

$$\begin{aligned}\omega_1 &= E^*[\mu_2^i - P_2] = \sqrt{\frac{2}{\pi}} \times \frac{n\sigma_\delta}{K_2} > 0,\\ \omega_{1Q} &= Var^*[\mu_2^i - P_2] = \frac{n^2}{K_2^2}(1 - \frac{2}{\pi})\sigma_\delta^2 > 0,\end{aligned}$$

where $K_2^i = (Var[\mu|\theta, \alpha^i, m])^{-1} = K_1 + n$, $K_2 = \int_0^1 K_2^i di = K_1 + n$, $\mu_2^i \equiv E[\mu|\theta, \alpha^i, m] = \frac{n}{K_2}(\theta - m_2^i)$, $\mu_2 \equiv E[\mu|\theta, m] = \frac{n}{K_2}(\theta - m)$, $\mu_1 = 0$, $\sigma_i^2 \equiv Var[P_2 - P_1|\alpha^i, m] = \frac{n}{K_1 K_2}$, $\sigma^2 \equiv Var[P_2 - P_1|m] = \frac{n}{K_1 K_2}$, *and* $\sigma_\delta^2 \equiv 1/\left(\frac{n^2}{K_2} + \frac{1}{\sigma_\epsilon^2}\right)$.

Proof. See Appendix A.1.

This theorem shows that the equilibrium prices are determined as if by the average agent in Period 2. Because the security and option have zero supply, the average agent requires no risk premium for holding the security and the option. In addition, difference-of-opinion about the public signal's mean in Period 2 only induces trading in the security but not in the option in Period 2, because agents agree on the precision of the security's liquidation value but disagree on its mean ($K_2^i = K_2$). Simple calculations show that when $\alpha^i < 0$ ($\alpha^i > 0$), $\mu_2^i > \mu_2$ ($\mu_2^i < \mu_2$), leading to $D_2^i > 0$ ($D_2^i < 0$); i.e., because the security price is determined by the average agent, an agent in Period 2, who is optimistic (pessimistic) about the signal, overestimates (underestimates) the expected payoff than the average agent, and thus buys (sells) the security.

Our primary interest is about the effect of subjective model uncertainty on the asset prices in Period 1. Suppose that there exists no uncertainty, so that agents know their subjective models in Period 1. Following a similar procedure as for Theorem 1, it can be proved that

$$P_1 = \mu_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1}.$$

Clearly, the prices of the risky assets in Period 1 are also determined by the average agent in this case (the proofs are available upon request). Therefore, ω_1 and ω_{1Q} are indeed induced by uncertainty about agents' subjective models. Standard finance models with uncertainty show that investors generally demand a uncertainty premium or price discount to hold assets with uncertain returns, i.e., investors dislike uncertainty—even beyond their dislike of risk.

Following the literature, we define the uncertainty premia implied by the security and option as $-\omega_1$ and $-\omega_{1Q}$, respectively. We use these notations throughout the paper.

Theorem 1 shows that subjective model uncertainty, more precisely, the uncertainty of agents' optimism (pessimism) about the public signal, induces negative uncertainty premia in both the security and option. In other words, subjective model uncertainty causes both the security and option to be more expensive.

The intuition is as follows. If the agents know their subjective models in Period 1, the prices are determined by the average agent based on his information set. Since he does not observe the public signal, the security price simply equals to the unconditional expectation of its liquidation value. In the presence of uncertainty, the agents are homogeneous in Period 1. Since the prices are determined by the average agent in Period 2, an agent believes that as long as he becomes a non-average agent later, he can take advantage of the average agent due to the difference-of-opinion about the signal mean. Because the agents are competitive, they will pay more than the average agent for the security in Period 1. Specifically, the amount of money he would pay equals the expected profit margin under the risk-neutral measure $E^*[\mu_2^i - P_2]$.

Furthermore, because of the uncertainty of the expected profit margin, an agent facing subjective model uncertainty will also pay more than the average agent for the option in Period 1. In particular, as the uncertainty level of agents' optimism (pessimism) σ_α increases, an agent facing subjective model uncertainty expects to profit more from the dispersion of interpretation in Period 2 and thus would pay more money for the option in Period 1.

We next use the results obtained in Theorem 1 to determine implied variance, variance risk premium, and trading volume. As in Heston (1993) and Bollerslev, Tauchen, and Zhou (2009), implied variance is defined as the return variance calculated under the risk-neutral measure. We summarize the results about implied variances in the following proposition.

Proposition 1. *The implied variance (the return variance calculated under the risk-neutral measure "*") in Period 2 is given by*

$$IV_2 \equiv Var_2^*[\mu - P_2] = Var_2^a[\mu - P_2] = \frac{1}{K_2}, \tag{10}$$

and the implied variance in Period 1 satisfies

$$IV_1 \equiv Var^*[P_2 - P_1] = \sigma^2 + \omega_{1Q}. \tag{11}$$

Proof. See Appendix A.2.

The expected variances EV_1 and EV_2 are calculated from the true distribution of the security returns

$$EV_2 \equiv Var^r[\mu - P_2] = \frac{1}{K_2}, \tag{12}$$

$$EV_1 \equiv Var^r[P_2 - P_1] = \sigma^2 + Var\big[E[P_2 - P_1|m]\big] = \sigma^2. \tag{13}$$

Since the prices are determined as if by the average agent, and the realized belief and the average agent's subjective belief about the signal mean are the same, the (average agent's) subjective variance and expected variance are the same—as in the stochastic volatility model with a representative agent who does not distinguish between subjective and realized volatilities.

Following the literature, we define the variance risk premium (VRP) as the difference of return variances between the risk-neutral measure and the physical or realized measure. From (10) and (13), $IV_2 = EV_2 = 1/K_2$. Hence, the variance risk premium in Period 2 is zero. Our main concern is the variance risk premium in Period 1, since our focus is the relationship between subjective model uncertainty and VRP. Using (11) and (13), we obtain

$$VRP \equiv IV_1 - EV_1 = \omega_{1Q} = \frac{n^2}{K_2^2}(1 - \frac{2}{\pi})\sigma_\delta^2 > 0. \tag{14}$$

Empirical evidence has shown that implied variance is a persistent process. Hence, we consider the percentage change in implied variance, DIV, which is more suitable in empirical testing. Using Proposition 1, we obtain that

$$DIV \equiv \frac{IV_2}{IV_1} = \frac{1}{K_2(\sigma^2 + \omega_{1Q})}. \tag{15}$$

Given that agents are homogeneous in Period 1, the trading volume V_1 is zero. Using agents' optimal security holdings in (8), the trading volume in Period 2 is determined by

$$V_2 = \int |D_2^i| di = \tau n \int |\alpha^i| di = \sqrt{\frac{2}{\pi}} \tau n \sigma_\alpha, \tag{16}$$

where τ is the risk tolerance coefficient.

As expected, difference-of-opinion about the signal mean induces trading in Period 2. Interestingly, (16) shows that trading volume does not depend on the public information, and thus trades occur without price change. The reason is that as the value of public information θ changes, because the agents (including the average agent) agree on the signal precision, their estimates of the security's liquidation value are updated by the same amount $n\theta/K_2$, and thus their trades remain the same. Kandel and Pearson (1995) obtain similar result in a model with two agents who have different opinions about the signal mean. Also, (16) shows that trading volume depends on two unobservable variables—n (average signal precision) and σ_α (variation in agents' dispersion of beliefs or uncertainty level of agents' optimism). Surprisingly, we can show that the trading volume can be written as functions of two empirically observable variables—variance risk premium (VRP) and percentage change in implied variance (DIV). The results are summarized in the following proposition.

Proposition 2. *In the presence of subjective model uncertainty about signal mean, trading volume V_2 is an increasing function of lagged VRP and a decreasing function of contemporaneous DIV, and is independent of contemporaneous absolute return (or realized volatility) $|P_2 - P_1|$.*

Proof. See Appendix A.3.

We next examine the case of belief uncertainty about the signal precision and show that the results are dramatically different.

2.3 Case II: Belief Uncertainty about Signal Precision

In this subsection, we examine the model of subjective model uncertainty about the precision of a public signal. All agents agree on the mean of the signal in Period 2, which is normalized to be zero ($m_2^i = 0$). In particular, agent i's belief regarding the signal precision n_2^i is given by

$$n_2^i = \nu \times n_c \times y^i, \tag{17}$$

where n_c and $y^i, i \in [0,1]$, are random variables and independent of each other; and ν is a positive constant reflecting the normalization factor. Here, n_c and y^i can be interpreted as the common and idiosyncratic components of agent i's belief about the signal precision, respectively. Denote agent i's confidence level ρ^i by $\rho^i = n_2^i/n$. The average agent's confidence level is $\rho^a = 1$.

The (log) common component $x_c \equiv \log(n_c)$ and the (log) idiosyncratic component $\epsilon_i \equiv \log(y^i)$ of agent i's belief about the signal precision follow normal distributions with mean 0 and variances of σ_n^2 and σ_ϵ^2, respectively. Then it follows that agent i's belief regarding the signal precision follows a lognormal distribution.[3] Using Central Limit Theorem, the average agent's belief of the signal precision in Period 2 satisfies:

$$n \equiv \int_0^1 n_2^i di = \nu n_c \exp\left(\frac{\sigma_\epsilon^2}{2}\right), \tag{18}$$

and agent i's confidence level in Period 2 is given by

$$\rho^i = \frac{y^i}{\exp\left(\frac{\sigma_\epsilon^2}{2}\right)}. \tag{19}$$

Put differently, agent i's confidence level in Period 2, ρ^i, depends only on the idiosyncratic component of his belief. By calculation, $Var(\rho^i) = \exp(\sigma_\epsilon^2) - 1$. Thus, σ_ϵ measures the uncertainty level of agents' confidence. As $Var(n_c)$ is an increasing function of σ_n, σ_n measures the uncertainty level of the average agent's signal precision.

To simplify notations, we set $\nu = \exp(-\sigma_\epsilon^2/2)$. Using (18), we obtain that

$$n_2^i = n_c \times \rho^i, \qquad n_c = n. \tag{20}$$

That is, agent i's belief regarding the signal precision is simply the product of the average agent's belief and agent i's confidence level.

As shown later in Theorem 2, the security and option prices are determined by the average agent once the agents observe the signal. Hence, it is sensible for us to assume that the realized precision of the signal equals the subjective belief of the average agent regarding

[3] It is noteworthy that the assumption of lognormal distribution is standard in stochastic volatility models with a representative agent, as in Gallant, Hsieh, and Tauchen (1997) and Taylor (2005).

the signal precision, that is,

$$n_2^r = n, \tag{21}$$

where (log) realized signal precision, $\epsilon_r = \log(n_2^r)$, follows a normal distribution with mean 0 and variance σ_n^2. We also assume that n_2^i and n_2^r are independent of θ. The equilibrium is solved with backwardation. The results are summarized in the following Theorem.

Theorem 2. *In the economy with subjective model uncertainty about the signal precision in Period 1, there exists an equilibrium in which prices are given by*

$$P_2 = \mu_2, \quad P_{2Q} = P_2^2 + \frac{1}{K_2}, \tag{22}$$

$$P_1 = \mu_1 + \omega_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1} + \omega_{1Q}, \tag{23}$$

the demands of agent i for the security and option are given by

$$D_{2Q}^i = \frac{\tau}{2}(1-\rho^i)n, \quad D_2^i = \tau(\mu_2^i - P_2)K_2^i - 2P_2 D_{2Q}^i, \tag{24}$$

$$D_{1Q}^i = 0, \quad D_1^i = 0, \tag{25}$$

and the uncertainty premia for the security and option in Period 1, $-\omega_1$ and $-\omega_{1Q}$, satisfy

$$\omega_1 = 0, \quad \omega_{1Q} = -\frac{E\left[\frac{\sqrt{K_2^i}n(n_2^i-n)}{K_2(K_1n_2^i+n^2)\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}\right]}{E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}\right]} = E^*[\frac{1}{2C_1^i} - \sigma^2], \tag{26}$$

where $K_2^i = (Var[\mu|\theta, n_2^i])^{-1} = K_1 + n_2^i$, $K_2 = \int_0^1 K_2^i di = K_1 + n$, $\mu_2^i = \frac{n_2^i}{K_2^i}\theta$, $\mu_2 = \frac{n}{K_2}\theta$, $\mu_1 = 0$, $\sigma_i^2 \equiv Var[P_2 - P_1|n_2^i, n] = \frac{K_2^i n^2}{K_1 K_2^2 n_2^i}$, $\sigma^2 \equiv Var[P_2 - P_1|n] = \frac{n}{K_1K_2}$, and $C_1^i = \frac{1}{2\sigma_i^2} + \frac{(\sigma^2-\sigma_i^2)^2 K_2^i}{2\sigma_i^2} \times \frac{K_1}{2n^2}(K_1 n_2^i + n^2)$.

Proof. See Appendix B.1.

Parameters K_2^i and K_2 represent the subjective precision levels of agent i and the average agent about the final payoff in Period 2, respectively. Equation (22) shows that the security and option prices in Period 2 are determined as if by the average agent. In Period 2, the option price is positively related to the squared security price. In addition, it is positively related to the conditional volatility based on the average agent's information set in Period 2.

Because agents are homogeneous in Period 1, there is no trading either in the security or the option, and agents' holdings are zero. In Period 2, because agent i is risk averse ($\tau > 0$), his demand for the security is positively related to his expected return ($\mu_2^i - P_2$) and his subjective precision (K_2^i) of the final payoff. Due to the demand to hedge his speculative trading in option, his security holdings are also negatively related to his positions in the option, ($-2P_2D_{2Q}^i$), or practically termed as "delta hedging". In Period 2, agent i trades the option speculatively, and his demand for option depends on his confidence level, $(1-\rho^i)$, i.e., whether he is more confident or less confident than the average agent. If $\rho^i > 1$, agent i has a high confidence level. Since he believes that the average agent overestimates the return volatility, he would hold a short position in the option. Similarly, if $\rho^i < 1$, agent i has a low confidence level. As he believes that the average agent underestimates the return volatility, he would hold a long position in the option.

Suppose that there exists no uncertainty about the signal precision, such that agents know their subjective models in Period 1. It can be easily shown that

$$P_1 = \mu_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1}.$$

The risky asset prices in Period 1 are also determined by the average agent in this case (the proofs are available upon request). Therefore, ω_{1Q} is induced by uncertainty about agents' subjective models.

Interestingly, the uncertainty premium embedded in the security price, $-\omega_1$, equals zero in Period 1 in the presence of uncertainty. As shown in Appendix B.1, this result holds for the more general case in which $x \neq 0$.[4] The reason for this result is that, as long as agents know their beliefs in Period 1, they agree on the expected return in Period 2, that is, $E[P_2 - P_1|\rho^i, n] = E[P_2 - P_1|n]$. Consequently, agents require no additional uncertainty premium in Period 1, even if they do not observe the public signal.

Equation (26) shows that the Period-1 uncertainty premium in the option, ω_{1Q}, takes a more complicated form than in the case of subjective model uncertainty about signal mean. It equals the difference between the expected values of σ^2 and the expected value of $1/(2C_2^i)$

[4]Similar calculations show that $\omega_1 = 0$ for the case that the agents are not allowed to trade the option.

under the risk neutral measure. Here, σ^2 measures the average agent's Period-2 subjective risk, $Var[P_2 - P_1|n]$. In contrast, $1/(2C_2^i)$ measures the non-average agent i's Period-2 subjective risk with a downward adjustment ($1/(2C_2^i) < \sigma_i^2$). The reason for the downward adjustment is that, agent i perceives that he would profit from the difference between his valuation of the security and the market price in Period 2 (or the average agent's valuation of the security), such that he faces less risk. Clearly, $1/(2C_2^i)$ incorporates both the common and idiosyncratic components of agents' belief about signal precision. In particular, when there is no confidence uncertainty ($n = n_2^i$) hence $1/(2C_2^i) = \sigma^2$, all agents are the same, and there is no uncertainty premium embedded in the option prices ($\omega_{1Q} = 0$).

In contrast, confidence uncertainty—the variation in dispersion of agents' beliefs—is crucial in generating the uncertainty premium implied in option price in our model. To better understand the option-implied uncertainty premium, $-\omega_{1Q}$, we first consider an approximate solution for the case of a small K_1—small prior precision hence large prior uncertainty, in which we are able to obtain a semi-closed form solution. This simplification also imitates the real world scenario that there is usually large prior uncertainty before a major economic news announcement. For tractability, we also assume that average uncertainty (σ_n) and idiosyncratic uncertainty (σ_ϵ) are small. Proposition 3 summarizes the solution.

Proposition 3. *In the economy with uncertainty about agents' beliefs regarding signal precision in Period 1, when σ_n, σ_ϵ, and K_1 are small, $\omega_{1Q} > 0$ and increases with σ_ϵ. More specifically, we have*

$$\omega_{1Q} \approx \frac{K_1 \sigma_\epsilon^2 \exp\{2\sigma_n^2\}}{(1 - \frac{3}{8}\sigma_\epsilon^2)} > 0. \tag{27}$$

Proof. See Appendix B.2.

This proposition shows that when the common prior precision (K_1) of the security payoff is small and that the confidence level uncertainty is small (then $\sigma_\epsilon^2 < \sqrt{8/3}$), option-implied uncertainty premium $-\omega_{1Q}$ is negative. The prices of the option are determined as if by the average agent in Period 2. When agents differ in their confidence levels, they trade the option to exploit the difference between their confidence levels and that of the average agent.

In Period 1, an agent believes that he can take advantage of the average agent whenever his confidence level differs from that of the average agent in Period 2. Consequently, he would like to pay more to hold the option ($\omega_{1Q} > 0$).

Proposition 1 also shows that ω_{1Q} depends on both σ_ϵ and σ_n. In particular, ω_{1Q} is positively related to σ_ϵ. As confidence level uncertainty increases, the non-average agent would earn a higher expected profit due to a larger difference of confidence level from that of the average agent in Period 2. Hence, agents would pay more for the option in Period 1, leading to a positive relationship between ω_{1Q} and σ_ϵ. We have also conducted extensive numerical exercises for general parameter values, and found that ω_{1Q} increases with σ_ϵ. This result is available upon request. Figures 1 and 2 later illustrate these relationships in simulation with calibrated parameter values.

We next use the results obtained in Theorem 2 to determine the expressions for implied variance, variance risk premium, and trading volume. The results about implied variances are summarized in the following proposition.

Proposition 4. *The implied variance (the return variance calculated under the risk-neutral measure "*") in Period 2 is given by*

$$IV_2 \equiv Var_2^*[\mu - P_2] = Var_2^a[\mu - P_2] = \frac{1}{K_2}, \tag{28}$$

and the implied variance in Period 1 satisfies the following equation

$$\begin{aligned} IV_1 &\equiv Var^*[P_2 - P_1] \\ &= \frac{E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2}(C_1^i)^{(3/2)}}\exp\left\{\frac{K_2-K_2^i}{2K_2}\right\}\right]}{2E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\left\{\frac{K_2-K_2^i}{2K_2}\right\}\right]} = E^*[\frac{1}{2C_1^i}] = E^*[\sigma^2] + \omega_{1Q}. \end{aligned} \tag{29}$$

Proof. See Appendix B.3.

Equation (29) shows that the implied variance $Var^*[P_2 - P_1]$ comprises two components: (1) $E^*\big[Var[P_2 - P_1|n]\big] = E^*[\sigma^2]$ represents the expected value of the return variance $Var[P_2 - P_1|n]$ conditional on the average agent's belief about the signal precision under the risk-neutral measure; and (2) ω_{1Q} represents the negative value of the uncertainty premium.

The expected variances EV_1 and EV_2 are calculated from the true distribution (or realized distribution) of the security returns:

$$EV_2 \equiv Var^r[\mu - P_2] = \frac{1}{K_2}, \tag{30}$$

$$EV_1 \equiv Var^r[P_2 - P_1] = E\left[\sigma^2\right] = E\left[\frac{n}{K_1 K_2}\right]. \tag{31}$$

Equations (28) and (31) show that $IV_2 = EV_2 = 1/K_2$. Hence, the variance risk premium in Period 2, which equals $IV_2 - EV_2$, is zero. Using (29) and (31), the variance risk premium in Period 1 is given by

$$VRP \equiv IV_1 - EV_1 = \left(E^*\left[\sigma^2\right] - E\left[\sigma^2\right]\right) + \omega_{1Q}. \tag{32}$$

A subtle point here is that the variance risk premium in our model could be driven by a risk-adjustment component ($E^*\left[\sigma^2\right] - E\left[\sigma^2\right]$), as in Bollerslev, Tauchen, and Zhou (2009), and/or a pure uncertainty component (ω_{1Q}), which does not depend on the risk tolerance coefficient τ at all. The novelty of our framework is that, even if we shut down the risk-adjustment channel, the uncertainty premium still remains the same, due to the subjective model uncertainty of agents' confidence levels.

Given the expression for implied variances in (28) and (29), the change in implied variances, DIV, can be easily shown as

$$DIV \equiv \frac{IV_2}{IV_1} = \frac{2E[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2 C_1^i}}\exp\{\frac{K_2 - K_2^i}{2K_2}\}]}{K_2 E[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2}(C_1^i)^{(3/2)}}\exp\{\frac{K_2 - K_2^i}{2K_2}\}]}. \tag{33}$$

Since agents are homogeneous in Period 1, the trading volume V_1 is zero. Given agents' optimal security holdings in (24), simple calculations shows that the trading volume in Period 2 is determined by

$$V_2 = \frac{1}{2}\int_0^1 |D_2^i| di = \frac{1}{2}\tau n \int_0^1 |\rho^i - 1||\theta| di. \tag{34}$$

In other words, trading volume in Period 2 depends (positively) on the risk tolerance coefficient (τ), the average signal precision (n), agents' confidence level relative to that of the average agent ($\rho^i - 1$), and the public signal (θ).

We have shown that the uncertainty of agents' confidence level is crucial to explain the seemingly overpriced options prices. We next show that the uncertainty level about confidence levels is crucial to generate trading volume and a non-zero variance risk premium in our model. Since the calculations are straightforward for the case without confidence uncertainty ($\sigma_\epsilon = 0$), we state the results without proofs in the following Proposition.

Proposition 5. *In the presence of subjective model uncertainty about signal precision, suppose that agents have homogeneous beliefs ($\sigma_\epsilon = 0$) in Period 2, then*

$$\begin{aligned} IV_1 &= EV_1 = E[\frac{n}{K_1 K_2}], \quad DIV = \frac{1}{K_2 E[\frac{n}{K_1 K_2}]}, \\ VRP &= \omega_{1Q} = 0, \quad \int_0^1 |\rho^i - 1| di = 0, \quad V_2 = 0. \end{aligned}$$

This proposition shows that when agents have homogeneous beliefs in Period 2, that is, $\sigma_\epsilon = 0$, implied variance IV_1 equals expected variance EV_1 in Period 1. As a result, both the uncertainty premium and variance risk premium are zero. Not surprisingly, agents do not trade in the absence of difference-of-opinion in Period 2.

Recall that in the case of subjective model uncertainty about the signal mean, the trading volume V_2 can be written as a function of two observable variables—DIV and VRP. We next show that in the case of subjective model uncertainty about the signal precision, we can also express the trading volume as a function of three observable variables. Simple calculations yield the following results.

Proposition 6. *The trading volume is given by*

$$V_2 = \tau \times |P_2 - P_1| \times \frac{NDO}{DIV}. \tag{35}$$

where $NDO = \frac{\int_0^1 |\rho^i - 1| di}{IV_1} = \frac{2[\Phi(\frac{\sigma_\epsilon^2}{2}) - \Phi(-\frac{\sigma_\epsilon^2}{2})]}{VRP + EV_1}$.

Proof. See Appendix B.4.

The trading volume contains three key components, in addition to the risk tolerance coefficient. The first term $|P_2 - P_1|$ is related to the realization of the public signal θ. Using Theorem 2, we obtain $\theta = \frac{K_2}{n}(P_2 - P_1)$. As in Bessembinder, Chan, and Seguin (1996) and Anderson, Bollerslev, Diebold, and Vega (2007), this term can be interpreted as a proxy of information flow. Our model predicts that more trades always occur as new information arrives (or as the security price changes). The second term DIV is related to the realization of agents' beliefs about the precision of the public information, reflecting the change in subjective risk (the average agent's return variances under the risk-neutral measure). Our model predicts that trading volume decreases with the average agent's subjective risk, because a higher subjective risk shall reduce the trading needs of the risk-averse agents. The third term NDO measures the normalized variation in difference-of-opinion ($\int_0^1 |\rho^i - 1| di / IV_1$). Our model predicts that the trading volume increases with this term, as a higher level of difference-of-opinion (after normalization) shall generate more speculative trading.

Both variance risk premium (VRP) and normalized difference-of-opinion (NDO) are functions of the uncertainty parameters σ_ϵ and σ_n, which are the main interest of this paper. We argue that VRP is a proxy for NDO under certain conditions. Since we are unable to derive closed-form expressions for VRP and NDO, we use simulation to illustrate their relationship. We first calibrate these two key parameters to obtain a benchmark model. Risk aversion is not a crucial parameter, as it does not affect VRP and the implied variances. We set $\tau = 1000$, which implies that agents are not very risk-averse. We also set $K_1 = 11$, $\sigma_n = 1.86$, and $\sigma_\epsilon = 2.83$, such that the implied expected variances (and thus VRP) roughly match those of S&P 500 in our empirical sample examined in Section 3.[5]

Figures 1 and 2 show the comparative statics results on variance risk premium (VRP) and the normalized measure of difference-of-opinion (NDO), concerning a changes in uncertainty of agents' belief in signal precision (σ_ϵ) and uncertainty of the average agent's belief in signal precision (σ_n), respectively. In simulation, we generate a random sample of $25,000,000$ observations to approximate the expectations.

Figure 1 shows that an increase in σ_ϵ leads to an increase in VRP, an increase in ω_{1Q}, and a decrease in NDO. On the other hand, Figure 2 shows that an increase in σ_n leads to a decrease in VRP, a decrease in ω_{1Q}, an increase in NDO. Figures 1 and 2 verify the results about ω_{1Q} in Proposition 3 for more general parameter values, that is, ω_{1Q} is positive and increases with σ_ϵ. These figures also show that VRP is positive and that its properties

[5]These parameters produce implied and expected variances as 20.23% and 12.5%, respectively.

of VRP are largely driven by ω_{1Q}.

In summary, Figures 1 and 2 suggest that under the reasonable parameter settings we consider here, there exists a negative relation between VRP and NDO, and thus a negative relationship between VRP and trading volume.

2.4 Model Comparisons and Empirical Implications

Given the equilibrium results in Sections 2.2 and 2.3, we are now ready to compare the equilibrium properties of the two models on subjective model uncertainty. We summarize the similarities and differences of these two models' properties in Table 1.

Subjective model uncertainty about signal mean and about signal precision yield many similar equilibrium properties. First, dispersion of beliefs generates trading in the security in Period 2. Second, there exists an option-implied uncertainty premium. In other words, the option is more expensive than without uncertainty. This prediction is consistent with the empirical finding that options prices tend to be overpriced compared with rational models based on risk premium, as in Constantinides, Czerwonko, Jackwerth, and Perrakis (2011). Third, subjective model uncertainty is an endogenous approach to model VRP. In contrast, typical stochastic volatility model, e.g., Heston (1993), is an exogenous approach to generate VRP. Fourth, both models of subjective model uncertainty generate positive variance risk premium, consistent with the recent empirical findings (see, e.g., Bollerslev, Tauchen, and Zhou, 2009; Drechsler and Yaron, 2011, among others). However, both uncertainty premium and VRP are independent of agents' risk aversion, which is in sharp contrast to the representative agent approaches on VRP, such as Heston (1993) and Bollerslev, Tauchen, and Zhou (2009). Fifth, VRP is generated by the uncertainty about non-average agents' types—the level of optimism or the level of confidence—rather than the uncertainty about the average agent's belief, even though the prices are determined as if by the average agent.

On the other hand, subjective model uncertainty about signal mean or about signal precision also generates significantly different results. First, the former predicts that in Period 2 agents disagree on expected returns but agree on return variances ($\mu_2^i \neq \mu_2$ and $K_2^i = K_2$), while the latter predicts that agents disagree on both expected returns and return variances ($\mu_2^i \neq \mu_2$ and $K_2^i \neq K_2$), due to different belief updating intensities. Second, the former predicts uncertainty premia in both the security and option ($\omega_1 > 0$ and $\omega_{1Q} > 0$), while the latter only predicts a uncertainty premium in the option ($\omega_1 = 0$ and $\omega_{1Q} > 0$). Third, the former predicts that option price and VRP are spanned by the security price, since the factors affecting the security price also affects the option price, while the latter predicts that VRP and options price are not spanned completely by the security price, since σ_ϵ affects VRP and the option price but not the security price. Several papers, e.g., Collin-Dufresne and Goldstein (2002), Carr and Wu (2009), and Constantinides, Czerwonko, Jackwerth, and Perrakis (2011), have documented evidence that VRP and options are not completely spanned by risk factors related to underlying securities. Fourth, the former predicts trades only in the security, while the latter predicts trades in both the security and option. Fifth, the former predicts that trades in the security can occur without price change and is always positively related to lagged VRP (controlling for current-period DIV), while the latter predicts that trading in the security is positively related to current-period absolute return and is negatively related to lagged VRP (controlling for current-period DIV).

Some of the differences have been documented in previous empirical work, as mentioned above. However, our model in the previous subsections generates novel and testable empirical hypotheses about the relationships between the security trading volume, lagged VRP, and volatility variables:

1. There exists a negative (positive) relationship between trading volume V_t and the lagged VRP, if there is subjective model uncertainty about signal precision (mean).

2. There exists a positive correlation between trading volume V_t and the absolute return (or realized volatility) $|P_t - P_{t-1}|$, if there is subjective model uncertainty about signal precision; while the correlation is zero, if there is subjective model uncertainty about signal mean.

Since both models also predict that there exists a negative relation between trading volume V_t and contemporaneous change in implied variances ($DIV_t = IV_t/IV_{t-1}$), we include DIV mainly as a control variable in the multivariate regression tests.

3 Empirical Evidence

In this section, we use futures market data of stock index, bond, and currency to test our model' implications of subjective model uncertainty for the relationship between trading volume and volatility risk premium. In general, information asymmetry plays a much less important role in these market compared with individual stocks. These markets are more likely driven by agents' different beliefs regarding information related to macroeconomy and public news announcements. Hence, they provide a good field experiment for testing our model's hypotheses. For these futures markets, there are corresponding options markets. As discussed before, in our framework, subjective model uncertainty about signal mean does not generate trading in options markets. We choose the futures markets to compare the predictions of the two models of subjective model uncertainty on trading volume. In the empirical part, we first use the volatility measures rather than the variance measures, because results of the former are more robust and less subject to potential outlier problems in the data. We then conduct robustness check with variance risk premium.

We next describe the data, the construction of expected volatility and volatility risk premium, the empirical specification, descriptive statistics, and empirical results.

3.1 Data

We obtain futures market data from Bloomberg, which includes three bond futures: U.S. 10-year treasury bond (TY), German 10-year government bond (RX), and Japan 10-year government bond (JB); three stock index futures: S&P500 index (SP), German DAX index (GX), and Nikkei index (NK); and three currency futures: Australian Dollar (AD), Euro (EC), and Japanese Yen (JY). Each futures contract consists of daily trading volumes, open interests, and prices with quarterly expiration dates from October 2006 to December 2016. We construct futures' volume and open interest series by adding up the nearest two contracts to expiration date.[6] We focus on the nearest two contracts because our model is about speculative trading and speculators usually trade short-term and active contracts.

[6]Note also that there are two different futures on S&P500 index: "SP" and "ES" (mini). We construct a composite volume index to represent the total trading volume of S&P500 futures.

We obtain implied volatility data from following sources. We use VIX from CBOE and VDAX from Deutsche Börse for the U.S. (SP) and German (GX) stock indices. We use VXJ for the Japanese (NK) stock index.[7] Bloomberg provides implied volatilities for currency futures of Euro (EC), Australian Dollar (AD), and Japanese Yen (JY). To be consistent with the stock index futures markets, we choose one month maturity for currency futures implied volatilities. For the 10-year bond futures of U.S. (TY), Germany (RX), and Japan (JB), we also use the one-month implied volatility indices of the nearest contract from Bloomberg.

3.2 Realized Volatility Forecast and Volatility Risk Premium

As in Anderson, Bollerslev, Diebold, and Vega (2007), we use the contract nearest to expiration to calculate returns r_t, since this contract is generally the most actively traded one; and when there is a contract rollover, we use the nearest two contracts to calculate the returns. Following Bessembinder, Chan, and Seguin (1996), Anderson, Bollerslev, Diebold, and Vega (2007), among others, we use daily absolute return as a measure of daily realized volatility $\sqrt{RV_t} = |r_t|$, where r_t denotes the return on day t. The normalized monthly and weekly realized volatilities are corresponding averages of daily realized volatilities.

Following Corsi (2009) and Drechsler and Yaron (2011), we use the daily, weekly and monthly realized volatilities plus the daily implied volatility to estimate the heterogeneous autoregressive model of realized volatility (HAR-RV). More specifically, we estimate the following model:

$$\sqrt{RV_{t+1}} = \alpha + \beta_D \sqrt{RV_t} + \beta_W \sqrt{RV_{t,week}} + \beta_M \sqrt{RV_{t,mon}} + \beta_V \sqrt{IV_t} + \epsilon_{t+1}, \quad (36)$$

where $\sqrt{IV_t}$ denotes the implied volatility—the square-root of implied variance.

The volatility risk premium is defined as the difference between the expected future volatilities under the risk-neutral and actual probability measures:

$$VRP_t \equiv E_t^*(\sqrt{RV_{t+1}}) - E_t(\sqrt{RV_{t+1}}), \quad (37)$$

where $E_t(\sqrt{RV_{t+1}})$ is estimated from Equation (36), and $E_t^*(\sqrt{RV_{t+1}}) = \sqrt{IV_t}/100/\sqrt{252}$.

[7] VXJ is constructed with the same methodology as VIX by the Center for the Study of Finance and Insurance. Details can be found at http://www-csfi.sigmath.es.osaka-u.ac.jp/en/activity/vxj.php.

3.3 Empirical Specification

To test the hypotheses regarding trading volume and volatility variables of our model, we estimate the following regression equation:

$$\begin{aligned} DTV_t &= c + \beta_1 DIV_t + \beta_2 VRP_{t-1} + \beta_3 ADR_t \\ &+ \beta_4 DTV_{t-1} + \beta_5 DTV_{t-2} + \beta_6 ADOI_{up,t} + \beta_7 ADOI_{dn,t} \\ &+ \beta_8 DTE_t + \beta_9 LST_t + \sum_{i=1}^{4} \gamma_i WD_{i,t} + \epsilon_t, \end{aligned} \tag{38}$$

where c is a constant and ϵ_t is the error term. We obtain coefficients by Ordinary Least Squares (OLS) estimation. To account for the potential heterogeneity and autocorrelation in the error term, we use Newey-West method to calculate the robust standard error with the lag length equal to $0.75N^{(1/3)}$, as suggested by Stock and Watson (2002). DTV_t represents daily log-transformed trading volume de-trended by its 62-day moving average. DIV_t denotes the implied volatility change and VRP_t denotes the volatility risk premium. Our model predicts a negative relation between trading volume and the percentage change in implied volatility ($\beta_1 < 0$), and a positive (negative) relation between trading volume and volatility risk premium [$\beta_2 > 0$ ($\beta_2 < 0$)] and a zero (positive) relation between trading volume and contemporaneous absolute return [$\beta_3 = 0$ ($\beta_3 > 0$)] if there exists subjective model uncertainty about signal mean (precision).

We also include standard control variables as in literature. Since trading volume is persistent, we expected both β_4 and β_5 to be positive. $ADOV_{up}$ equals the absolute change in open interest on the days when open interest increases and 0 otherwise, $ADOV_{dn}$ equals the absolute change in open interest on the days when open interest decreases and 0 otherwise. Bessembinder, Chan, and Seguin (1996) suggest that open interest may be a good proxy for difference of opinion. Following their paper, we expect the estimate of β_6 to be positive and significantly larger than that of β_7. DTE is the days to expiration dummy, and LST denotes the last trading day dummy. We expect β_8 and β_9 to be negative, because trading diminishes when approaching maturity. WD_i, $i \in [1, 2, 3, 4]$, denote the day-of-week dummies.

3.4 Descriptive Statistics

Table 2 reports the summary statistics for the nine futures contracts. Panel A shows the de-trend log-transformed trading volumes (i.e., the trading volume subtracted by the time trends, which are calculated as its 62-day moving average). Two stylized facts emerge. First, trading volumes have negative skewness and large varying kurtosis, deviating from the normal distributions. Second, trading volumes are persistent series, as the first-order autocorrelation coefficients of trading volumes are between 0.44 and 0.79 at daily frequency.

Panel B of Table 2 reports the summary statistics of daily implied volatility, which is quoted in annualized percentage. The first-order autocorrelation coefficients for implied volatilities are all larger than 0.94. Implied volatilities show positive skewness and large kurtosis, suggesting asymmetry and right tail. The means and medians of implied volatilities of stock index futures are all above 19.6, larger than those of currency futures (all around 10), which in turn are larger than those of bond futures (all around 5). Such differences of implied volatilities among different asset classes could also be driven—at least partially—by the asset class differences of volatility risk premia, which we turn to next.

Panel C of Table 2 reports descriptive statistics of daily volatility risk premium. The first-order autocorrelation coefficients are smaller than those of the corresponding implied volatilities, but some are still larger than 0.9. Volatility risk premia show positive skewness and large kurtosis, indicating right fat tails. It is also interesting to note that, for volatility risk premia, both means and medians of equity index futures (around 5-8) are much larger than those of currency futures (around 2-3), which in turn are larger than those of bond futures (around 1-2).

3.5 Empirical Findings

We present the main empirical results in Table 3. Our aim is to see whether the empirical relationships conform to the model's implied ones as stated in Hypotheses 1 and 2. Columns 2 through 10 in Table 3 report the estimates for each futures with all standard control variables. In addition to the coefficient estimates and Newey-West standard errors, we also report R^2s and adjusted R^2s.

Table 3 reports the results on lagged volatility risk premium, the absolute return, and the contemporaneous implied volatility change. The marginal effects of volatility risk premium on the trading volume are uniformly negative and mostly significant at 5 percent level or higher, except for Japan stock index futures (NK) that is not statistically significant and for U.S. stock index futures (SP) that is only significant at 10 percent level. We find that the marginal effects of absolute return are uniformly significant at 1 percent level. The marginal effects of the change in implied volatility on the trading volume are uniformly positive and remain statistically significant at 5 percent level or higher. In terms of coefficient magnitudes, the lagged volatility risk premia have a range of -0.113 to -0.004, the contemporaneous absolute return has a range of 0.047 to 0.171, and the contemporaneous implied volatility changes have a range of 0.161 to 1.315. In other words, one percentage increase in implied volatility or absolute return or variance risk premium results in 0.161 to 1.315 percentage increase, 0.047 to 0.171 percentage increase, or 0.004 to 0.113 percentage decrease in trading volume, respectively. Overall, the regression R^2s or adjusted R^2s are in a high range of 0.53 to 0.69.

For robustness, we also tests the same specification in the forms of variance risk premium (Table 4) and log variance risk premium (Table 5), which can negatively predict trading volume in three bond futures and three currency futures markets (mostly significant) but positively predict trading volume in three stock index futures markets (about half insignificant).[8]

To summarize, the empirical results are more consistent with the implications of subjective model uncertainty about signal precision rather than about signal mean. It is noteworthy that the finding that trading volume is positively related to contemporaneous absolute return does not necessarily mean there exists no effect of uncertainty about signal mean, because

[8] Intuitively, these findings are consistent with the notion that government bond and foreign exchange markets mainly reflect discount rate (volatility) uncertainty, while stock markets reflect both discount rate (volatility) uncertainty and cash-flow (drift) uncertainty. We leave this for future research. Here, variance risk premium and log variance risk premium are defined by $E_t^*(RV_{t+1}) - E_t(RV_{t+1})$ and $\log(E_t^*(RV_{t+1})) - \log(E_t(RV_{t+1}))$, respectively, where $E_t^*(RV_{t+1}) = IV_t/10000/252$ and $E_t(RV_{t+1})$ is estimated from $RV_{t+1} = \alpha + \beta_D RV_t + \beta_W RV_{t,week} + \beta_M RV_{t,mon} + \beta_V IV_t + \epsilon_{t+1}$.

we cannot exclude completely other effects, such as asymmetric information (see, e.g., Kyle, 1985). We also find that trading volume is positively related to the contemporaneous implied volatility change, inconsistent with both models' prediction. We consider this as a puzzle, since it is intuitive that an increase in the average agent's subjective risk shall reduce the trading of risk-averse agents, and leave it for future research.

4 Conclusion

In this paper, we analyze a three-period model on subjective model uncertainty. The agents trade in both a security and an option. The security's liquidation value is realized in Period 3. The agents observe a public signal in Period 2 and interpret the signal with different subjective models. In Period 1, they are homogeneous and uncertain about some of their model parameter—signal mean or signal precision.

Belief uncertainty about either signal mean or signal precision produces a genuine negative uncertainty premium component in the option price, which results from agents' initial uncertainty about their optimism level or confidence level and is independent of agents' risk attitude. This uncertainty premium causes the option to be more expensive in Period 1. As a result, subjective model uncertainty endogenously generates positive variance risk premium (VRP).

However, subjective model uncertainty about signal mean or signal precision also produces dramatically different results. First, the former produces uncertainty premia in the security and option, while the latter only produces a uncertainty premium in the option. Second, the former indicates that the option prices, the uncertainty premium, and VRP are spanned completely by the security prices, while the latter indicates the opposite. Third, the former suggests trading only in the security, while the latter suggests trading in both the security and option. Forth, the former implies that trades in the security occur without price change and is always positively related to lagged VRP, while the latter implies that trading volume in the security is positively related to current-period absolute return and can be negatively related to lagged VRP.

We empirically test the two models' implications about the relationships between trading volume and volatility (variance) risk premium, absolute return, and implied volatility. The empirical results using data from major futures markets of stock index, bonds, and foreign exchanges largely confirm the predictions of the model on subjective model uncertainty about signal precision.

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A Proofs regarding Subjective Model Uncertainty about Signal Mean

In this Appendix, we presents the proofs for the case in which there exists subjective model uncertainty about signal mean. We assume that agents know and agree on the signal precision. We first prove Theorem 1. We then use the results in Theorem 1 to prove Proposition 1 and Proposition 2.

A.1 Proof of Theorem 1

We solve the equilibrium with backwardation. In Period 2, agents observe the public signal and know their own beliefs about signal mean. Conjecture that the prices of the security and option in Period 2 are given by

$$P_2 = \mu_2, \quad P_{2Q} = P_2^2 + \frac{1}{K_2}, \tag{39}$$

where μ_2 and K_2 are defined in Theorem 1.

The terminal wealth of agent i is given by $W_3^i = W_2^i + D_2^i(\mu - P_2) + D_{2Q}^i(\mu^2 - P_{2Q})$. By calculation, agent i's optimization problem is given by

$$\begin{aligned}
V_2^i &= \max_{\{D_{2Q}^i, D_2^i\}} E_2^i[U(W_3^i)] \\
&\propto -\sqrt{K_2^i}\int_{-\infty}^{\infty} \exp\{-\frac{W_2^i + D_2^i(\mu - P_2) + D_{2Q}^i(\mu^2 - P_{2Q})}{\tau} - \frac{K_2^i(\mu - \mu_2^i)^2}{2}\}d\mu, \\
&\propto -\sqrt{K_2^i}\int_{-\infty}^{\infty} \exp\left[-\frac{W_2^i - D_{2Q}^i P_{2Q} + P_2^2 D_{2Q}^i}{\tau} - \frac{(\mu_2^i - P_2)^2 K_2^i}{2} - (\frac{D_{2Q}^i}{\tau} + \frac{K_2^i}{2})(\mu - P_2)^2\right] \\
&\quad \times \exp\left[-(\frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - \frac{2(\mu_2^i - P_2)K_2^i}{2})(\mu - P_2)\right]d\mu, \\
&\propto -\frac{\sqrt{K_2^i}}{\sqrt{C_2^i}}\exp\left[\frac{(S_2^i)^2}{4C_2^i} - \frac{W_2^i - D_{2Q}^i P_{2Q} + P_2^2 D_{2Q}^i}{\tau} - \frac{(\mu_2^i - P_2)^2 K_2^i}{2}\right],
\end{aligned} \tag{40}$$

where $S_2^i = \frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - (\mu_2^i - P_2)K_2^i$ and $C_2^i = \frac{D_{2Q}^i}{\tau} + \frac{K_2^i}{2}$ and the expressions for μ_2, μ_2^i, K_2^i, and K_2. The first order conditions with respect to D_2^i and D_{2Q}^i are given by

$$\begin{aligned}
\frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - (\mu_2^i - P_2)K_2^i &= 0, \\
-\frac{1}{2C_2^i\tau} + \frac{P_{2Q} - P_2^2}{\tau} &= 0.
\end{aligned}$$

Using $K_2^i = K_2$ and $P_2 = \mu_2$, Simplification gives

$$D_{2Q}^i = 0, \qquad D_2^i = \tau(\mu_2^i - \mu_2)K_2 = -\tau n\alpha^i. \tag{41}$$

Note that if we replace "i" by "a", then we obtain the optimal demands for the average agent. For the average agent, we know that $K_2^a = K_2$ and $\mu_2^a = 0$. Plugging them into (41), we obtain $D_{2Q}^a = 0$ and $D_2^a = x$. Imposing the market clearing conditions show that the conjectured forms of the prices of the security and option are satisfied. Plugging the optimal positions of agent i back into (40) gives

$$V_2^i \propto -\exp\{-\frac{W_2^i}{\tau} - \frac{K_2(\mu_2^i - P_2)^2}{2}\}. \tag{42}$$

Given agent i's optimal trading and value function in Period 2, we next derive his optimal trading in Period 1. We conjecture that the prices in Period 1 are given by

$$P_1 = \mu_1 + \omega_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1} + \omega_{1Q}, \tag{43}$$

where ω_1 and ω_{1Q} represent the uncertainty premia in the security and option, respectively. Because agents are homogeneous, for simplicity, we use $E[.]$ to denote the expectation based on the information set of agent i in Period 1.

The wealth of agent i in Period 2 is given by $W_2^i = W_1^i + D_1^i(P_2 - P_1) + D_{1Q}^i(P_{2Q} - P_{1Q})$. Substituting agent i's wealth into his utility function, using the conjectured price functions, and taking expectation, agent i's optimization problem is equivalent to

$$\begin{aligned} & \max_{\{D_{1Q}^i, D_1^i\}} \quad E[U(W_3^i)] = E[V_2^i] \\ = \; & E\left[-\exp\{-\frac{W_1^i + D_1^i(P_2 - P_1) + D_{1Q}^i(P_{2Q} - P_{1Q})}{\tau} - \frac{K_2(\mu_2^i - P_2)^2}{2}\}\right]. \end{aligned}$$

We first calculate $E[V_2^i|m, \alpha^i]$. Define σ^2 as the conditional return variance from Period 1 to Period 2 conditional on m. By calculation, $\sigma^2 \equiv Var[(P_2 - P_1)|m] = \frac{n}{K_1 K_2}$. Conditional on m and α^i, $P_2 - P_1$ follows a normal distribution. Simple calculation shows that $(P_2 - P_1) = \frac{n}{K_2}(\theta - m) - \omega_1$. Hence, we have

$$\begin{aligned} e_1^i &\equiv E[(P_2 - P_1)|m, \alpha^i] = \frac{n\alpha^i}{K_2} - \omega_1, \\ e_1 &\equiv E^a[(P_2 - P_1)] = E[(P_2 - P_1)|m] = -\omega_1, \\ \sigma_i^2 &\equiv Var[(P_2 - P_1)|m, \alpha^i] = \sigma^2. \end{aligned}$$

By calculation, we have

$$\begin{aligned} \mu_2^i - P_2 &= A_1^i(P_2 - P_1) + A_2^i(\mu_1 - P_1) + A_3^i, \\ P_{2Q} - P_{1Q} &= (P_2 - P_1)^2 + 2P_1(P_2 - P_1) - \frac{n}{K_2 K_1} - \omega_{1Q}, \end{aligned}$$

where $A^i_1 = 0$, $A^i_2 = 0$, and $A^i_3 = -\frac{n}{K_2}\alpha^i$. Note that $\mu^i_2 - P_2$ does not depend on the public signal θ and thus the Period-2 return $P_2 - P_1$.

Define $Z \equiv P_2 - P_1$. Plugging the conjectured price functions and the wealth process into $E[V^i_2|m, \alpha^i]$ yields

$$\begin{aligned} V^i_1 \quad &\equiv \quad E[V^i_2|m,\alpha^i] \propto \int_{-\infty}^{\infty} -\sqrt{\frac{1}{\sigma^2}} \exp \{-\frac{D^i_{1Q}Z^2 + (2P_1 D^i_{1Q} + D^i_1)Z}{\tau} - \frac{(-\sigma^2 - \omega_{1Q})D^i_{1Q}}{\tau}\} \\ &\times \exp\{-\frac{K_2[A^i_1 Z + A^i_2(\mu_1 - P_1) + A^i_3]^2}{2} - \frac{(Z - e^i_1)^2}{2\sigma^2}\}dZ, \\ \propto \quad &\int_{-\infty}^{\infty} -\sqrt{\frac{1}{\sigma^2}} \exp\left[-C^i_1[Z + \frac{S^i_1}{2C^i_1}]^2 + H^i\right] dZ \propto -\sqrt{\frac{1}{\sigma^2 C^i_1}} \exp\{H^i\}, \end{aligned}$$

where

$$\begin{aligned} C^i_1 \quad &= \quad \frac{D^i_{1Q}}{\tau} + \frac{1}{2\sigma^2}, \qquad S^i_1 = \frac{2P_1 D^i_{1Q} + D^i_1}{\tau} - \frac{e^i_1}{\sigma^2}, \\ H^i \quad &= \quad \frac{\sigma^2 + \omega_{1Q}}{\tau} D^i_{1Q} - \frac{K_2 (A^i_3)^2}{2} - \frac{(e^i_1)^2}{2\sigma^2} + \frac{(S^i_1)^2}{4C^i_1}. \end{aligned}$$

By the law of iterated expectations, agent i's optimization problem in Period 1 is thus given by

$$\max_{\{D^i_1, D^i_{1Q}\}} \quad E\big[E[U(W^i_3)|m, \alpha^i]\big] = E[V^i_1].$$

Taking first order derivatives with respect to D^i_1 and D^i_{1Q} yields

$$E\left[-\frac{1}{\tau\sigma_i\sqrt{C^i_1}} \exp\{H^i\} \times \frac{S^i_1}{2C^i_1}\right] = 0, \tag{44}$$

and

$$E\left(-\frac{1}{\tau\sqrt{\sigma^2 C^i_1}} \exp\{H^i\} \left[-\frac{1}{2C^i_1} + (\sigma^2 + \omega_{1Q}) + \frac{S^i_1 P_1}{C^i_1} - (\frac{S^i_1}{2C^i_1})^2\right]\right) = 0. \tag{45}$$

Because agents do not know their beliefs regarding the precision of the public signal in Period 1, there exists no trading. Market clearing conditions imply

$$D^i_1 = 0, \quad D^i_{1Q} = 0. \tag{46}$$

In equilibrium, using these expressions, we obtain

$$2\sigma^2 C^i_1 = 1, \quad \frac{S^i_1}{C^i_1} = 2e^i_1, \quad H^i = -\frac{n^2(\alpha^i)^2}{2K_2}.$$

Substituting them into (44) and (45) and simple calculations give

$$E\left[\exp\{H^i\}\alpha^i\right] = \sqrt{\frac{2}{\pi}} \times \frac{\sigma_\delta^2}{\sigma_\alpha}, \qquad E\left[\exp\{H^i\}\right] = \frac{\sigma_\delta}{\sigma_\alpha},$$
$$E\left[\exp\{H^i\}(\alpha^i)^2\right] = \frac{\sigma_\delta^3}{\sigma_\alpha},$$

where $\sigma_\delta^2 = 1/\left(\frac{n^2}{K_2} + \frac{1}{\sigma_\alpha^2}\right)$.

Submitting these expression into (44) and (45) and yields the solutions to ω_1 and ω_{1Q}:

$$\omega_1 = \frac{n}{K_2} \times \frac{E\left[\exp\{H^i\}\alpha^i\right]}{E\left[\exp\{H^i\}\right]} = \sqrt{\frac{2}{\pi}} \times \frac{n\sigma_\delta}{K_2} > 0, \tag{47}$$

$$\omega_{1Q} = \frac{n^2}{K_2^2} \times \frac{E\left[\exp\{H^i\}(\alpha^i)^2\right]}{E\left[\exp\{H^i\}\right]} - \omega_1^2 = \frac{n^2}{K_2^2}(1 - \frac{2}{\pi})\sigma_\delta^2 > 0, \tag{48}$$

The uncertainty premia can also be expressed as forms under the risk neutral world, which will be shown in the proof of Proposition 1.

A.2 Proof of Proposition 1

We use Theorem 1 to derive the implied volatilities in this model. We first determine $IV_2 \equiv E^*[(\mu - P_2)^2]$, which is the expected variation of the final payoff under the risk-neutral measure. To this end, we need to determine the stochastic discount factor in Period 2 and thus the risk neutral measure. Recall that the stochastic discount factor is calculated based on the utility of the representative agent, whose holdings in the security and option equal the asset supplies. In our model, the average agent serves as the representative agent. The FOCs of the average agent are given by

$$E_2^a[U(W_3^a)(\mu - P_2)] = 0, \qquad E_2^a[U(W_3^a)(\mu^2 - P_{2Q})] = 0,$$

where W_3^a is calculated at the optimal holdings of the average agent.

Define $M_2 \equiv \frac{U(W_3^a)}{E_2^a[U(W_3^a)]}$. It is straightforward to show that

$$P_2 = E_2^a[M_2\mu], \quad P_{2Q} = E_2^a[M_2\mu^2].$$

Under the risk neutral world, we have

$$P_2 = E_2^*[(\mu - P_2)], \qquad P_{2Q} = E_2^*[\mu^2].$$

where "*" denotes the risk-neutral measure. Because $E_2^a[M_2] = 1$, we know that M_2 is the stochastic discount factor in Period 2. Simple calculations yield

$$Var_2^*[\mu - P_2] = E_2^a[M_2(\mu - P_2)^2] = E_2^a\left[M_2[\mu^2 - P_{2Q} + \frac{1}{K_2} - 2P_2(\mu - P_2)]\right]$$
$$= \frac{E_2^a[M_2]}{K_2} = \frac{1}{K_2} = Var_2^a[(\mu - P_2)^2].$$

Hence, the implied volatility of the security return in Period 2 is simply the average agent's belief about return variation under the original actual measure.

We then solve for $IV_1 \equiv E^*[(P_2 - P_1)^2]$. Investors are homogeneous and know only the distributions of their beliefs in Period 1. Hence, we can calculate the discount factor based on the utility of any agent i. Define $M_1 \equiv \frac{V_2^i}{E[V_2^i]}$, where V_2^i is calculated at agent i's optimal holdings in the security and option in Period 1. By calculation, we obtain that $E[M_1] = 1$. The FOCs of agent i can be rewritten as

$$E[V_2^i(P_2 - P_1)] = 0, \qquad E[V_2^i(P_{2Q} - P_{1Q})] = 0,$$

where the expected utility of agent i, V_2^i, is calculated at the optimal holdings in the security and option. By calculation, we have

$$P_1 = E[M_1 P_2], \quad P_{1Q} = E[M_1 P_{2Q}].$$

Under the risk neutral world, we have

$$P_1 = E^*[P_2], \qquad P_{1Q} = E_2^*[P_{2Q}].$$

As a result, M_1 is the stochastic discount factor in Period 1. Simple calculations give

$$IV_1 = E^*[(P_2 - P_1)^2] = E\left[M_1[(P_{2Q} - P_{1Q}) - 2P_1(P_2 - P_1) + \frac{n}{K_1 K_2} + \omega_{1Q}\right] = \sigma^2 + \omega_{1Q},$$

where $\sigma^2 = \frac{n}{K_1 K_2}$.

Plugging the expression of M_1 into (47) and using the law of iterated expectations, we can expression the uncertainty premia in the risk neutral world:

$$\begin{aligned}
\omega_1 &= E\left[\frac{V_2^i}{E[V_2^i]} e_1^i\right] = E^*[\frac{n}{K_2}\alpha^i] = E^*[\mu_2^i - P_2], \\
\omega_{1Q} &= E\left[\frac{V_2^i}{E[V_2^i]} (e_1^i)^2\right] = \frac{n}{K_2}\{E^*[(\alpha^i)^2] - \left(E^*[\alpha^i]\right)^2\} \\
&= Var^*[\frac{n}{K_2}\alpha^i] = Var^*[\mu_2^i - P_2].
\end{aligned}$$

When agents agree on the signal precision, the uncertainty premia in the security and option are simply equal to the mean and variance of agent i's profit margin (per share) $\mu_2^i - P_2$ under the risk neutral measure.

A.3 Proof of Proposition 2

Rearranging (15) and (14) yields

$$\begin{aligned} n &= \frac{1/DIV - VRP \times K_1}{VRP + \frac{1}{K_1}}, \\ n^2\sigma_\alpha^2 &= \frac{K_2^2}{(\frac{1-2/\pi}{VRP} - K_2)} = \frac{\left(1+\frac{1}{DIV}\right)^2}{\left(1+\frac{1}{K_1 VRP}\right)\left[\frac{\pi-2}{\pi K_1} - (\frac{2}{\pi}+\frac{1}{DIV})VRP\right]}. \end{aligned} \tag{49}$$

It can be seen that $n^2\sigma_\alpha^2$ is a decreasing function of DIV and an increasing function of VRP. Using (16), V_2 decreases with DIV and increases with VRP.

B Proofs regarding Subjective Model Uncertainty about Signal Precision

In this Appendix, we presents the proofs for the case in which there exists subjective model uncertainty about signal precision. We assume that agents know and agree on the signal mean. We first prove Theorem 2. We then use the results in Theorem 2 to prove Proposition 3, Proposition 4, and Proposition 6. The proofs are similar to those in Appendix A.

B.1 Proof of Theorem 2

We solve the equilibrium for the general case in which $x \neq 0$ in the presence of subjective model uncertainty about signal precision. In Period 2, agents observe the public signal and know their own beliefs. We conjecture that the prices of the security and option in Period 2 are given by

$$P_2 = \mu_2 - \frac{x}{\tau K_2}, \quad P_{2Q} = P_2^2 + \frac{1}{K_2}. \tag{50}$$

The terminal wealth of agent i is given by $W_3^i = W_2^i + D_2^i(\mu - P_2) + D_{2Q}^i(\mu^2 - P_{2Q})$. By calculation, agent i's optimization problem is given by

$$\begin{aligned} V_2^i &= \max_{\{D_{2Q}^i, D_2^i\}} E_2^i[U(W_3^i)] \\ &\propto -\sqrt{K_2^i}\int_{-\infty}^{\infty} \exp\{-\frac{W_2^i + D_2^i(\mu - P_2) + D_{2Q}^i(\mu^2 - P_{2Q})}{\tau} - \frac{K_2^i(\mu-\mu_2^i)^2}{2}\}d\mu, \\ &\propto -\sqrt{K_2^i}\int_{-\infty}^{\infty} \exp\left[-\frac{W_2^i - D_{2Q}^i P_{2Q} + P_2^2 D_{2Q}^i}{\tau} - \frac{(\mu_2^i - P_2)^2 K_2^i}{2} - (\frac{D_{2Q}^i}{\tau} + \frac{K_2^i}{2})(\mu - P_2)^2\right] \\ &\quad \times \exp\left[-(\frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - \frac{2(\mu_2^i - P_2)K_2^i}{2})(\mu - P_2)\right]d\mu, \end{aligned}$$

$$\propto -\frac{\sqrt{K_2^i}}{\sqrt{C_2^i}} \exp\left[\frac{(S_2^i)^2}{4C_2^i} - \frac{W_2^i - D_{2Q}^i P_{2Q} + P_2^2 D_{2Q}^i}{\tau} - \frac{(\mu_2^i - P_2)^2 K_2^i}{2}\right], \tag{51}$$

where $S_2^i = \frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - (\mu_2^i - P_2)K_2^i$ and $C_2^i = \frac{D_{2Q}^i}{\tau} + \frac{K_2^i}{2}$. The first order conditions with respect to D_2^i and D_{2Q}^i are given by

$$\frac{2D_{2Q}^i P_2 + D_2^i}{\tau} - (\mu_2^i - P_2)K_2^i = 0, \qquad -\frac{1}{2C_2^i \tau} + \frac{P_{2Q} - P_2^2}{\tau} = 0.$$

Simplification gives

$$D_{2Q}^i = \frac{\tau}{2}(K_2 - K_2^i), \qquad D_2^i = \tau(\mu_2^i - P_2)K_2^i - 2P_2 D_{2Q}^i. \tag{52}$$

For the average agent, we know that $K_2^a = K_2$ and $\mu_2^a = \mu_2$. Plugging them into (52), we obtain $D_{2Q}^a = 0$ and $D_2^a = x$. Imposing the market clearing conditions show that the conjectured forms of the prices of the security and option are satisfied. Plugging the optimal positions of agent i back into (51) gives

$$V_2^i \propto -\frac{\sqrt{K_2^i}}{\sqrt{K_2}} \exp\{-\frac{W_2^i}{\tau} - \frac{K_2^i(\mu_2^i - P_2)^2}{2} + \frac{K_2 - K_2^i}{2K_2}\}. \tag{53}$$

Compared with (42), we have a new term $(K_2 - K_2^i)/(2K_2)$ for the case of subjective model uncertainty about signal precision.

Given agent i's optimal trading and value function in Period 2, we next derive his optimal trading in Period 1. We conjecture that the prices in Period 1 are given by

$$P_1 = \mu_1 - \frac{x}{\tau K_1} + \omega_1, \quad P_{1Q} = P_1^2 + \frac{1}{K_1} + \omega_{1Q}, \tag{54}$$

where ω_1 and ω_{1Q} represent the subjective model uncertainty premia in the security and option, respectively. As before, we use $E^i[.]$ to denote the expectation based on the information set of agent i in Period 1.

The wealth of agent i in Period 2 is given by $W_2^i = W_1^i + D_1^i(P_2 - P_1) + D_{1Q}^i(P_{2Q} - P_{1Q})$. Substituting agent i's wealth into his utility function, using the conjectured price functions, and taking expectation, agent i's optimization problem is equivalent to

$$\max_{\{D_{1Q}^i, D_1^i\}} \quad E[U(W_3^i)] = E[V_2^i]$$

$$= E\left[-\sqrt{\frac{K_2^i}{K_2}} \exp\{-\frac{W_1^i + D_1^i(P_2 - P_1) + D_{1Q}^i(P_{2Q} - P_{1Q})}{\tau} - \frac{K_2^i(\mu_2^i - P_2)^2}{2} + \frac{K_2 - K_2^i}{2K_2}\}\right].$$

We first calculate $E[V_2^i | n, n_2^i]$. Define σ^2 as the conditional return variance from Period 1 to Period 2 based on n (that is, assume that the average agent knows his belief). By calculation, $\sigma^2 \equiv Var[(P_2 - P_1)|n] = \frac{n}{K_1 K_2}$. Conditional on n and n_2^i, $P_2 - P_1$ follows a normal distribution. Simple calculation shows that $(P_2 - P_1) = \frac{n}{K_2}(\theta - \mu_1) + \frac{x}{\tau}\sigma^2 - \omega_1$. Hence, we have

$$e_1^i \equiv E[(P_2 - P_1)|n, n_2^i] = \frac{x}{\tau}\sigma^2 - \omega_1,$$
$$\sigma_i^2 \equiv Var[(P_2 - P_1)|n, n_2^i] = \frac{K_2^i n}{K_2 n_2^i}\sigma^2.$$

The agents agree on the mean of the signal. Hence, e_1^i depends only on the average agent's belief n and we simply denote it by e_1. By calculation, we have

$$\begin{aligned} \mu_2^i - P_2 &= A_1^i(P_2 - P_1) + A_2^i(\mu_1 - P_1) + A_3^i, \\ P_{2Q} - P_{1Q} &= (P_2 - P_1)^2 + 2P_1(P_2 - P_1) - \frac{n}{K_2 K_1} - \omega_{1Q}, \end{aligned}$$

where $A_1^i = (\frac{n_2^i}{K_2^i}\frac{K_2}{n} - 1) = (\sigma^2/\sigma_i^2 - 1)$, $A_2^i = \frac{K_1}{K_2^i}$, and $A_3^i = \frac{n_2^i K_1}{K_2^i n}\omega_1$.

Define $Z \equiv P_2 - P_1$. Plugging the conjectured price functions and the wealth process into $E[V_2^i|n_2^i, n]$ yields

$$\begin{aligned} V_1^i &\equiv E[V_2^i|n_2^i, n] \propto \int_{-\infty}^{\infty} -\sqrt{\frac{K_2^i}{K_2\sigma_i^2}} \exp\ \{-\frac{D_{1Q}^i Z^2 + (2P_1 D_{1Q}^i + D_1^i)Z}{\tau} - \frac{(-\sigma^2 - \omega_{1Q})D_{1Q}^i}{\tau}\} \\ &\times \exp\{-\frac{K_2^i[A_1^i Z + A_2^i(\mu_1 - P_1) + A_3^i]^2}{2} - \frac{(Z - e_1)^2}{2\sigma_i^2} + \frac{K_2 - K_2^i}{2K_2}\}dZ, \\ &\propto \int_{-\infty}^{\infty} -\sqrt{\frac{K_2^i}{K_2\sigma_i^2}} \exp\left[-C_1^i[Z + \frac{S_1^i}{2C_1^i}]^2 + H^i\right] dZ \propto -\sqrt{\frac{K_2^i}{K_2\sigma_i^2 C_1^i}} \exp\{H^i\}, \end{aligned}$$

where

$$C_1^i = \frac{D_{1Q}^i}{\tau} + \frac{1}{2\sigma_i^2} + \frac{(A_1^i)^2 K_2^i}{2}, \tag{55}$$

$$S_1^i = \frac{2P_1 D_{1Q}^i + D_1^i}{\tau} - \frac{e_1}{\sigma_i^2} + K_2^i A_1^i[A_2^i(\mu_1 - P_1) + A_3^i], \tag{56}$$

$$H^i = \frac{\sigma^2 + \omega_{1Q}}{\tau} D_{1Q}^i - \frac{K_2^i[A_2^i(\mu_1 - P_1) + A_3^i]^2}{2} - \frac{e_1^2}{2\sigma_i^2} + \frac{(S_1^i)^2}{4C_1^i} + \frac{K_2 - K_2^i}{2K_2}. \tag{57}$$

By the law of iterated expectations, agent i's optimization problem in Period 1 is thus given by

$$\max_{\{D_1^i, D_{1Q}^i\}} \quad E\big[E[U(W_3^i)|n_2^i, n]\big] = E[V_1^i].$$

Taking first order derivatives with respect to D_1^i and D_{1Q}^i yields

$$E\left[-\frac{\sqrt{K_2^i}}{\tau\sigma_i\sqrt{K_2 C_1^i}} \exp\{H^i\} \times \frac{S_1^i}{2C_1^i}\right] = 0, \tag{58}$$

and

$$E\left(-\frac{\sqrt{K_2^i}}{\tau\sigma_i\sqrt{K_2C_1^i}}\exp\{H^i\}\left[-\frac{1}{2C_1^i}+(\sigma^2+\omega_{1Q})+\frac{S_1^iP_1}{C_1^i}-(\frac{S_1^i}{2C_1^i})^2\right]\right)=0. \tag{59}$$

Because agents do not know their beliefs regarding the precision of the public signal in Period 1, there exists no trading. Market clearing conditions imply

$$D_1^i=x, \quad D_{1Q}^i=0. \tag{60}$$

In other words, agent i's holdings in the security and option in Period 1 equal x and 0, respectively. Solving (58) and (59) gives the solutions to ω_1 and ω_{1Q}. By calculation, we have

$$\begin{aligned}[A_2^i(\mu_1-P_1)+A_3^i] &= \frac{x}{\tau K_2^i}+\frac{(n_2^i-n)K_1}{K_2^i n}\omega_1,\\ \frac{e_1}{\sigma_i^2} &= \frac{K_2n_2^ix}{nK_2^i\tau}-(\frac{K_2}{n})^2\frac{n_2^iK_1}{K_2^i}\omega_1.\end{aligned}$$

Plugging (60) into the expression of S_1^i and using the expressions of A_1^i, A_2^i, P_1, and A_3^i yields

$$S_1^i=\frac{K_1}{K_2^in^2}\left[n_2^iK_2^2+(n_2^i-n)^2\right]\omega_1.$$

Using $n>0$, $K_2>0$, $n_2^i>0$, and $K_1>0$, (58) simplifies to

$$\omega_1=0, \quad S_1^i=0.$$

Thus, there is no uncertainty risk premium in the security price related to belief uncertainty regarding the precision of the public signal. To understand the intuition, setting $D_{1Q}^i=0$ in (56) and using $S_1^i=0$ yields

$$D_1^i=\frac{e_1}{\sigma_i^2}-K_2^iA_1^i[A_2^i(\mu_1-P_1)+A_3^i].$$

Simple calculation shows

$$\begin{aligned}H^i &= -\frac{x^2}{2K_2^i\tau^2}-\frac{n_2^ix^2}{2\tau^2K_1K_2^i}+\frac{K_2-K_2^i}{2K_2}\\ &= -\frac{x^2}{2\tau^2K_2^i}[1+\frac{n_2^i}{K_1}]+\frac{K_2-K_2^i}{2K_2}=-\frac{x^2}{2\tau^2K_1}+\frac{K_2-K_2^i}{2K_2},\\ C_1^i &= \frac{1}{2\sigma_i^2}+\frac{(\sigma^2/\sigma_i^2-1)^2K_2^i}{2}=\frac{K_1}{2n^2}(K_1n_2^i+n^2), \quad \sigma_i^2C_1^i=\frac{n^2K_2^i+n_2^iK_1K_2^i}{2K_2^2n_2^i}.\end{aligned}$$

Rearrangement of (59) yields

$$\omega_{1Q}=-\frac{E\left[\frac{\sqrt{K_2^i}n(n_2^i-n)}{K_2(K_1n_2^i+n^2)\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}\right]}{E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}\right]}. \tag{61}$$

As in the case of subjective model uncertainty about signal mean, the uncertainty premium ω_{1Q} can also be expressed as forms under the risk neutral world, which will be shown in the proof of Proposition 4.

B.2 Proof of Proposition 3

We prove that $\omega_{1Q} > 0$ when K_1 is small with Taylor series expansion. Using (26), since $E[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}} \exp\{\frac{K_2-K_2^i}{2K_2}\}] > 0$, we only need to show that

$$Q_\omega \equiv -E\left[\frac{\sqrt{K_2^i}n(n_2^i-n)}{K_2(K_1n_2^i+n^2)\sigma_i\sqrt{K_2C_1^i}} \exp\{\frac{K_2-K_2^i}{2K_2}\}\right] > 0. \tag{62}$$

We assume that σ_n^2 and σ_ϵ^2 are small so that Taylor series expansion can give us valid solution. Taylor expansion around $K_1 = 0$ yields

$$\begin{aligned}
&\exp\{\frac{K_2-K_2^i}{2K_2}\} \approx 1 + (K_2-K_2^i)/(2K_2) \approx (3-\rho^i)/2 - [(1-\rho^i)K_1]/(2n),\\
&\sqrt{K_2^i/K_2} \approx \sqrt{\rho^i}[1+K_1(1/n_2^i - 1/n)/2], \quad n/K_2 \approx 1 - K_1/n,\\
&n^2/(K_1n_2^i+n^2) \approx 1 - K_1n_2^i/n^2, \quad 1/(\sigma_i\sqrt{C_i}) \approx \sqrt{2}[1-(\frac{1}{n_2^i}+\frac{n_2^i}{n^2}-\frac{2}{n})\frac{K_1}{2}].
\end{aligned}$$

Plugging these expressions into (62) yields

$$\begin{aligned}
Q_\omega \;\approx\; & -\sqrt{2}E\Big[\tfrac{\sqrt{\rho^i}(\rho^i-1)}{n}(1-\tfrac{K_1}{n})[1+\tfrac{K_1}{2}(\tfrac{1}{n_2^i}-\tfrac{1}{n})](1-\tfrac{n_2^iK_1}{n^2})\\
&\cdot[\tfrac{3-\rho^i}{2}-\tfrac{(1-\rho^i)K_1}{2n}][1-(\tfrac{1}{n_2^i}+\tfrac{n_2^i}{n^2}-\tfrac{2}{n})\tfrac{K_1}{2}]\Big].
\end{aligned} \tag{63}$$

Note that ρ^i and n are independent of each other. By calculation, $E[(\rho^i)^k] = \exp\{k(k-1)/2\sigma_\epsilon^2\}$, where k is a constant. When σ_ϵ^2 is small, using Taylor series expansion, we obtain

$$\begin{aligned}
E[(\rho^i)^{-1/2}] \;&\approx\; 1+3\sigma_\epsilon^2/8, \quad E[(\rho^i)^{1/2}] \approx 1-\sigma_\epsilon^2/8, \quad E[(\rho^i)^{3/2}] \approx 1+3\sigma_\epsilon^2/8,\\
E[(\rho^i)^2] \;&\approx\; 1+\sigma_\epsilon^2, \quad E[(\rho^i)^{5/2}] \approx 1+15\sigma_\epsilon^2/8, \quad E[(\rho^i)^{7/2}] \approx 1+35\sigma_\epsilon^2/8.
\end{aligned}$$

Substituting them into (63) yields

$$\begin{aligned}
Q_\omega \quad &\approx \frac{\sqrt{2}K_1}{4}E\left[\sqrt{\rho^i}(\rho^i-1)[(3-\rho^i)(3\rho^i+1)+2-2\rho^i]\right]E\left[\frac{1}{n^2}\right]\\
&\approx \frac{\sqrt{2}K_1}{4}E\left[9(\rho^i)^{5/2}-3(\rho^i)^{7/2}-5(\rho^i)^{1/2}-(\rho^i)^{3/2}\right]E\left[\frac{1}{n^2}\right],\\
&\approx \sqrt{2}K_1\sigma_\epsilon^2E[\frac{1}{n^2}] > 0.
\end{aligned} \tag{64}$$

We next derive the semi-closed form solution for $E[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}]$. Similar calculations yield

$$
\begin{aligned}
& E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2-K_2^i}{2K_2}\}\right] \\
& \approx \frac{\sqrt{2}}{2}E\left[\sqrt{\rho^i}(3-\rho^i)+\frac{K_1\sqrt{\rho^i}}{n}[(3-\rho^i)(1-\rho^i)-(1-\rho^i)]\right], \\
& \approx \sqrt{2}(1-\frac{3}{8}\sigma_\epsilon^2). \qquad (65)
\end{aligned}
$$

Combining (64) and (65) yields

$$W_{1Q} \approx \frac{K_1\sigma_\epsilon^2\exp\{2\sigma_n^2\}}{(1-\frac{3}{8}\sigma_\epsilon^2)} > 0. \qquad (66)$$

It easy to see that W_{1Q} increases with σ_ϵ. In other words, the magnitude of the uncertainty premium increases with the uncertainty level of agents' confidence.

B.3 Proof of Proposition 4

Using Theorem 2, we first determine $IV_2 \equiv E^*[(\mu-P_2)^2]$, which is the expected variation of the final payoff under the risk-neutral measure. Similar to the case of subjective model uncertainty about signal mean, the average agent serves as the representative agent. The FOCs of the average agent are given by

$$E_2^a[U(W_3^a)(\mu-P_2)]=0, \qquad E_2^a[U(W_3^a)(\mu^2-P_{2Q})]=0,$$

where W_3^a is calculated at the optimal holdings of the average agent in the security and option.

Define $M_2 \equiv \frac{U(W_3^a)}{E_2^a[U(W_3^a)]}$. It is straightforward to show that

$$P_2 = E_2^a[M_2\mu], \quad P_{2Q} = E_2^a[M_2\mu^2].$$

Because $E_2^a[M_2]=1$, we know that M_2 is the stochastic discount factor in Period 2. Given the conjectured price functions, we obtain

$$
\begin{aligned}
Var_2^*[\mu-P_2] &= E_2^a[M_2(\mu-P_2)^2] = E_2^a\left[M_2[\mu^2-P_{2Q}+\frac{1}{K_2}-2P_2(\mu-P_2)]\right] \\
&= E_2^a[M_2]\frac{1}{K_2} = \frac{1}{K_2} = Var_2^a[(\mu-P_2)^2].
\end{aligned}
$$

Hence, the implied volatility of the security return in Period 2 is simply the average agent's belief about return variation under the original actual measure.

We then determine $IV_1 \equiv E^*[(P_2 - P_1)^2]$. To determine the risk-neutral measure, we first find the stochastic discount factor under the actual measure. Investors are homogeneous and know only the distributions of their beliefs in Period 1. Hence, we can calculate the discount factor based on the utility of any agent i. Define $M_1 \equiv \frac{V_2^i}{E[V_2^i]}$, where V_2^i is calculated at agent i's optimal holdings in the security and option in Period 1. By calculation, we obtain that $E[M_1] = 1$. The FOCs of agent i can be rewritten as

$$0 = E[V_2^i(P_2 - P_1)], \quad 0 = E[V_2^i(P_{2Q} - P_{1Q})],$$

where the expected utility of agent i, V_2^i, is calculated at the optimal holdings in the security and option. By calculation, we have

$$P_1 = E[M_1 P_2], \quad P_{1Q} = E[M_1 P_{2Q}].$$

Thus, M_1 is the stochastic discount factor in Period 1. Using (59) and (61), we obtain that

$$\begin{aligned}
IV_1 &= E^*[(P_2 - P_1)^2] = E\left[M_1[(P_{2Q} - P_{1Q}) - 2P_1(P_2 - P_1) + \frac{n}{K_1K_2} + \omega_{1Q}]\right] \\
&= E\left[M_1[\frac{n}{K_1K_2} + \omega_{1Q}]\right] = E\left[(\frac{n}{K_1K_2} + \omega_{1Q})E[M_1|n_2^i, n]\right] \\
&= \frac{E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2}(C_1^i)^{(3/2)}}\exp\{\frac{K_2 - K_2^i}{2K_2}\}\right]}{2E\left[\frac{\sqrt{K_2^i}}{\sigma_i\sqrt{K_2C_1^i}}\exp\{\frac{K_2 - K_2^i}{2K_2}\}\right]} = E^*[\frac{1}{2C_1^i}].
\end{aligned}$$

Plugging the expression of M_1 into (61) and using the law of iterated expectations yields

$$\omega_{1Q} = E^*[\frac{1}{2C_1^i} - \sigma^2].$$

B.4 Proof of Proposition 6

We first prove that

$$\int_0^1 |\rho^i - 1|di = E[|\rho^i - 1|] = 2[\Phi(\frac{\sigma_\epsilon^2}{2}) - \Phi(-\frac{\sigma_\epsilon^2}{2})], \tag{67}$$

where $\Phi(.)$ denotes the cumulative standard normal distribution. From the Central Limit Theorem, $\int_0^1 |\rho^i - 1|di = E[|\rho^i - 1|]$.

Define $o_i = \frac{1}{\sqrt{2\pi}\sigma_\epsilon}\left[\exp\{\epsilon_i - \frac{\sigma_\epsilon^2}{2}\} - 1\right] \times \exp\{\frac{\epsilon_i^2}{2\sigma_\epsilon^2}\}$. Simple calculations yield

$$E[|\rho^i - 1|] = \int_{\sigma_\epsilon^2/2}^{\infty} o_i d\epsilon_i - \int_{-\infty}^{\sigma_\epsilon^2/2} o_i d\epsilon_i, \tag{68}$$

By rearrangement, we have

$$\begin{aligned}\int_{\sigma_\epsilon^2/2}^{\infty} o_i d\epsilon_i &= \int_{\sigma_\epsilon^2/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\epsilon} \exp\{-\frac{(\epsilon_i - \sigma_\epsilon^2)^2}{2\sigma_\epsilon^2}\} - \int_{\sigma_\epsilon^2/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\epsilon} \exp\{-\frac{\epsilon_i^2}{2\sigma_\epsilon^2}\} \\ &= \Phi(\sigma_\epsilon^2/2) - \Phi(-\sigma_\epsilon^2/2). \end{aligned} \tag{69}$$

Similarly, we obtain

$$\int_{-\infty}^{\sigma_\epsilon^2/2} o_i d\epsilon_i = \Phi(\sigma_\epsilon^2/2) - \Phi(-\sigma_\epsilon^2/2). \tag{70}$$

Substituting (69) and (70) into (68) yields (67).

Substituting (33) and (70) into (34) and rearrangement gives

$$\begin{aligned} V_2 &= \tau K_2 \times |P_2 - P_1| \times [\Phi(\frac{\sigma_\epsilon^2}{2}) - \Phi(-\frac{\sigma_\epsilon^2}{2})], \\ &= = \tau |P_2 - P_1| \times DIV \times NDO. \end{aligned}$$

where $NDO = \frac{\int_0^1 |\rho^i - 1| di}{IV_1} = \frac{2[\Phi(\frac{\sigma_\epsilon^2}{2}) - \Phi(-\frac{\sigma_\epsilon^2}{2})]}{VRP + EV_1}$ and $\Phi(.)$ denotes the accumulated normal distribution.

Table 1: Subjective Model Uncertainty about Signal Mean and Precision

Panel A. Similarities

No.	Similar features
1	Trading in security is generated by dispersion of beliefs in Period 2
2	Options prices are more expensive than without uncertainty
3	VRP is endogenously induced by uncertainty about agents' types
4	VRP is positive but independent of risk aversion
5	VRP is due to uncertainty of non-average agents' types

Panel B. Differences

NO.	Uncertainty regarding signal mean	Uncertainty regarding signal precision
1	Agree on return variance and disagree on expected return	Disagree on expected return and return variance
2	Uncertainty premia in the security and option	Uncertainty premium only in the option
3	VRP and option prices are spanned completely by the security price	VRP and option prices are not spanned by the security price
4	Trades only occur in the security	Trades occur in the security and option
5	Security trading volume is positively related to lagged VRP	Security trading volume is negatively related to lagged VRP

Table 2: Summary Statistics of Volume, Implied Volatility, and Volatility Risk Premium

Panel A. Log levels of trading volumes

Assets	Number of obs.	Mean	Standard error	Skewness	Kurtosis	Median	AR(1)
TY	2654	13.88	0.53	-2.05	20.73	13.91	0.79
RX	2670	13.55	0.47	-0.34	5.13	13.54	0.68
JB	2576	10.30	0.44	0.20	3.91	10.26	0.68
SP	2661	14.43	0.75	-8.65	20.41	14.46	0.30
GX	2667	11.80	0.43	-0.13	3.88	11.82	0.72
NK	2577	11.24	0.53	-4.72	81.25	11.25	0.53
EC	2660	12.29	0.55	-5.55	74.40	12.32	0.44
AD	2660	11.23	0.68	-4.44	59.81	11.34	0.63
JY	2657	11.65	0.52	-3.46	51.39	11.65	0.51

Panel B. Implied volatility

Assets	Number of obs.	Mean	Standard error	Skewness	Kurtosis	Median	AR(1)
TY	2654	6.13	1.84	1.06	4.30	5.79	0.99
RX	2670	5.22	1.32	0.87	3.18	4.85	0.99
JB	2576	4.50	1.55	1.40	6.72	4.25	0.94
SP	2661	20.21	6.70	0.79	3.44	19.66	0.98
GX	2667	22.96	9.43	1.11	3.93	21.30	0.99
NK	2577	24.49	6.19	0.79	4.52	24.08	0.97
EC	2660	10.25	2.04	0.90	3.82	9.85	0.98
AD	2660	10.23	2.47	0.60	3.06	9.95	0.99
JY	2657	10.99	2.99	1.37	5.38	10.20	0.98

Panel C. Volatility risk premium

Assets	Number of obs.	Mean	Stared error	Skewness	Kurtosis	Median	AR(1)
TY	2654	1.761	0.875	1.017	4.348	1.566	0.964
RX	2670	1.432	0.61	0.843	3.323	1.282	0.956
JB	2576	1.314	0.606	1.746	10.855	1.222	0.773
SP	2661	7.81	1.505	0.894	4.013	7.641	0.909
GX	2667	5.938	1.556	1.146	5.562	5.645	0.565
NK	2577	7.96	1.602	0.959	5.472	7.75	0.838
EC	2660	2.985	0.792	0.916	4.015	2.862	0.954
AD	2660	2.49	0.829	0.695	3.615	2.383	0.947
JY	2657	2.898	0.977	1.003	4.796	2.703	0.767

Note: Panels A contains daily summary statistics of log-transformed trading volume from October 2006 to December 2016, which are stated in number of contracts, and are constructed by adding up generic "1" and "2" contracts. Panel B contains descriptive statistics of annualized implied volatilities, which are stated in percentage. Panel C contains descriptive statistics of volatility risk premium $VORP$, which is defined by $VORP_t \equiv E_t^*(\sqrt{RV_{t+1}}) - E_t(\sqrt{RV_{t+1}})$, where $E_t(\sqrt{RV_{t+1}})$ is estimated from the HAR-RV model in (36). $VORP$ is multiplied by $\sqrt{252}\times 100$ for illustration purpose.

Table 3: Volatility Risk Premium and Trading Volumes

	TY	RX	JB	SP	GX	NK	EC	AD	JY
Volatility risk premium	-0.068**	-0.084***	-0.051***	-0.066*	-0.040**	-0.004	-0.113***	-0.035***	-0.029***
	(0.026)	(0.018)	(0.014)	(0.041)	(0.022)	(0.016)	(0.017)	(0.006)	(0.005)
Absolute return	0.074***	0.071***	0.111***	0.047***	0.067***	0.061***	0.104***	0.081***	0.171***
	(0.008)	(0.006)	(0.010)	(0.010)	(0.008)	(0.011)	(0.008)	(0.012)	(0.012)
Control variables									
Implied volatility Change	0.418**	0.451**	0.161***	1.033***	1.315***	0.956***	0.409***	0.737***	0.982***
	(0.134)	(0.073)	(0.047)	(0.088)	(0.105)	(0.094)	(0.125)	(0.135)	(0.112)
Increase in open interest	-0.137	0.217	-0.361***	2.107*	0.127*	1.459***	2.438***	1.338***	1.275***
	(0.204)	(0.081)	(0.085)	(1.436)	(0.062)	(0.279)	(0.285)	(0.234)	(0.239)
Decrease in open interest	-0.247	0.013*	0.074	-0.622***	-0.060	-0.179**	0.124	0.036	0.079
	(0.172)	(0.070)	(0.056)	(0.221)	(0.084)	(0.075)	(0.118)	(0.097)	(0.104)
Volume: lag 1	0.459***	0.437***	0.432***	0.707***	0.445***	0.575***	0.398***	0.527***	0.387***
	(0.035)	(0.034)	(0.020)	(0.025)	(0.026)	(0.021)	(0.075)	(0.027)	(0.051)
Volume: lag 2	0.098***	0.161***	0.087***	-0.010	0.140***	0.054**	0.078***	0.063***	0.113***
	(0.025)	(0.019)	(0.017)	(0.013)	(0.019)	(0.019)	(0.019)	(0.021)	(0.025)
Days to expiration	-0.002***	-0.001***	-0.002***	-0.002***	-0.001***	-0.001***	-0.003***	-0.002**	-0.002***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Expiration indicator	-0.089**	-0.334***	-0.299	-0.075*	-0.187***	-0.570***	-0.144***	-0.151***	-0.188***
	(0.036)	(0.049)	(0.042)	(0.040)	(0.034)	(0.045)	(0.038)	(0.034)	(0.043)
R^2	0.554	0.635	0.606	0.695	0.593	0.652	0.535	0.577	0.579
$Adj.R^2$	0.552	0.633	0.604	0.694	0.591	0.649	0.532	0.574	0.576

Note: This table presents coefficients obtained from regression of futures log-transformed trading volumes on volatility risk premium, absolute return, and control variables at daily intervals. The sample period is from October 2006 to December 2016. Volume and open interest series are defined as their log levels de-trended by subtracting the 62-day moving averages. The change in implied volatility is defined in Equation (33). Volatility risk premium is defined by $VORP_t \equiv E_t^*(\sqrt{RV_{t+1}}) - E_t(\sqrt{RV_{t+1}})$, where $E_t(\sqrt{RV_{t+1}})$ is estimated from the HAR-RV model in (36)

$$\sqrt{RV_{t+1}} = \alpha + \beta_D \sqrt{RV_t} + \beta_W \sqrt{RV_{t,week}} + \beta_M \sqrt{RV_{t,mon}} + \beta_V \sqrt{IV_t} + \epsilon_{t+1},$$

and $E_t^*(\sqrt{RV_{t+1}}) = \sqrt{IV_t}/100/\sqrt{252}$. Open interest increase variable is equal to absolute change in open interest on the days when open interest increases and 0 otherwise. Open interest decrease variable equals the absolute change in open interest on the days when open interest decreases and 0 otherwise. Absolute return is the absolute value of futures return. Standard errors are given in parentheses. Coefficients significantly different from zero at 10 percent level, 5 percent level, and 1 percent level, are indicated by *, **, and ***, respectively. Both absolute return and variance risk premium are normalized by their sample means. The robust standard errors in parenthesis are calculated using Newey and West method. To save space, we omit the estimated coefficients for weekday dummies and the constant.

Table 4: Variance Risk Premium and Trading Volumes

	TY	RX	JB	SP	GX	NK	EC	AD	JY
Variance risk premium	-0.022**	-0.020***	-0.019***	0.015***	0.013***	0.003	-0.031***	-0.071***	-0.050***
	(0.007)	(0.004)	(0.004)	(0.004)	(0.003)	(0.003)	(0.005)	(0.009)	(0.009)
Absolute return	0.071***	0.072***	0.103***	0.049***	0.069***	0.056***	0.108***	0.079***	0.168***
	(0.008)	(0.007)	(0.009)	(0.008)	(0.008)	(0.010)	(0.009)	(0.011)	(0.012)
Control variables									
Implied volatility Change	0.421**	0.420**	0.185***	1.015***	1.297***	0.938***	0.344***	0.732***	0.966***
	(0.133)	(0.079)	(0.048)	(0.093)	(0.107)	(0.097)	(0.153)	(0.141)	(0.115)
Increase in open interest	-0.007	0.329**	-0.372***	2.70*	0.159*	2.133***	3.000***	1.916***	2.052***
	(0.230)	(0.151)	(0.086)	(1.668)	(0.098)	(0.311)	(0.310)	(0.225)	(0.252)
Decrease in open interest	-0.178	0.038	0.104	-0.580***	-0.056	-0.134**	0.202	0.087	0.181
	(0.167)	(0.078)	(0.06)	(0.217)	(0.084)	(0.073)	(0.149)	(0.098)	(0.113)
Volume: lag 1	0.560***	0.534***	0.522***	0.732***	0.499***	0.612***	0.407***	0.541***	0.418***
	(0.032)	(0.030)	(0.024)	(0.022)	(0.022)	(0.021)	(0.073)	(0.025)	(0.047)
Volume: lag 2	0.092***	0.157***	0.111***	-0.004	0.188***	0.052**	0.084***	0.062***	0.119***
	(0.025)	(0.019)	(0.017)	(0.013)	(0.019)	(0.019)	(0.02)	(0.02)	(0.026)
Days to expiration	-0.003***	-0.003***	-0.004***	-0.002***	-0.003***	-0.002***	-0.004***	-0.003**	-0.004***
	(0.0004)	(0.0003)	(0.0003)	(0.0004)	(0.0003)	(0.0002)	(0.0004)	(0.0003)	(0.0005)
Expiration indicator	-0.129**	-0.537***	-0.279	-0.115*	-0.292***	-0.142***	-0.194***	-0.199***	-0.260***
	(0.036)	(0.041)	(0.047)	(0.035)	(0.035)	(0.035)	(0.036)	(0.033)	(0.040)
R^2	0.513	0.568	0.556	0.689	0.565	0.629	0.416	0.569	0.508
$Adj.R^2$	0.510	0.568	0.556	0.687	0.563	0.627	0.413	0.567	0.505

Note: This table presents coefficients obtained from regression of futures log-transformed trading volumes on variance risk premium, absolute return, and control variables at daily intervals. The sample period is from October 2006 to December 2016. Volume and open interest series are defined as their log levels de-trended by subtracting the 62-day moving averages. The change in implied volatility is defined in Equation (33). Variance risk premium is defined by $VRP_t \equiv E_t^*(RV_{t+1}) - E_t(RV_{t+1})$, where $E_t(RV_{t+1})$ is estimated from the HAR-RV model

$$RV_{t+1} = \alpha + \beta_D RV_t + \beta_W RV_{t,week} + \beta_M RV_{t,mon} + \beta_V IV_t + \epsilon_{t+1}$$

and $E_t^*(RV_{t+1}) = IV_t/10000/252$. Open interest increase variable is equal to absolute change in open interest on the days when open interest increases and 0 otherwise. Open interest decrease variable equals the absolute change in open interest on the days when open interest decreases and 0 otherwise. Absolute return is the absolute value of futures return. Standard errors are given in parentheses. Coefficients significantly different from zero at 10 percent level, 5 percent level, and 1 percent level, are indicated by *, **, and ***, respectively. Both absolute return and variance risk premium are normalized by their sample means. The robust standard errors in parenthesis are calculated using Newey and West method. To save space, we omit the estimated coefficients for weekday dummies and the constant.

Table 5: Log Variance Risk Premium and Trading Volumes

	TY	RX	JB	SP	GX	NK	EC	AD	JY
Log variance risk premium	-0.022***	-0.015***	-0.012***	0.001	0.023	0.021***	-0.043***	-0.115***	-0.115***
	(0.007)	(0.005)	(0.005)	(0.004)	(0.015)	(0.007)	(0.007)	(0.03)	(0.038)
Absolute return	0.068***	0.066***	0.103***	0.031***	0.067***	0.055***	0.105***	0.107***	0.183***
	(0.008)	(0.006)	(0.010)	(0.007)	(0.008)	(0.010)	(0.009)	(0.007)	(0.013)
Control variables									
Implied volatility Change	0.442**	0.453**	0.189***	0.773***	1.291***	0.946***	0.390***	0.891***	1.030***
	(0.133)	(0.078)	(0.051)	(0.092)	(0.108)	(0.099)	(0.153)	(0.120)	(0.120)
Increase in open interest	-0.001	0.333**	-0.376***	6.52***	0.163	2.118***	2.946***	1.870***	2.039***
	(0.229)	(0.151)	(0.086)	(0.699)	(0.10)	(0.315)	(0.301)	(0.240)	(0.279)
Decrease in open interest	-0.184	0.037*	0.105	-0.547***	-0.050	-0.138*	0.199	0.078	0.187
	(0.168)	(0.080)	(0.060)	(0.260)	(0.082)	(0.074)	(0.149)	(0.100)	(0.123)
Volume: lag 1	0.561***	0.537***	0.528***	0.713***	0.502***	0.602***	0.407***	0.517***	0.408***
	(0.032)	(0.030)	(0.024)	(0.026)	(0.023)	(0.023)	(0.072)	(0.025)	(0.049)
Volume: lag 2	0.094***	0.159***	0.112***	-0.004	0.187***	0.059**	0.087***	0.058***	0.115***
	(0.025)	(0.019)	(0.017)	(0.012)	(0.019)	(0.022)	(0.019)	(0.019)	(0.027)
Days to expiration	-0.003***	-0.003***	-0.004***	-0.002***	-0.003***	-0.002***	-0.004***	-0.003**	-0.003***
	(0.0003)	(0.0003)	(0.00)	(0.0003)	(0.0003)	(0.0003)	(0.0007)	(0.0003)	(0.0006)
Expiration indicator	-0.127**	-0.545***	-0.282	-0.065*	-0.291***	-0.140***	-0.192***	-0.191***	-0.233***
	(0.036)	(0.042)	(0.048)	(0.035)	(0.035)	(0.035)	(0.037)	(0.034)	(0.041)
R^2	0.511	0.567	0.554	0.708	0.563	0.631	0.413	0.571	0.499
$Adj.R^2$	0.508	0.565	0.551	0.706	0.561	0.629	0.413	0.571	0.499

Note: This table presents coefficients obtained from regression of futures log-transformed trading volumes on the log variance risk premium, absolute return, and control variables at daily intervals. The sample period is from October 2006 to December 2016. Volume and open interest series are defined as their log levels de-trended by subtracting the 62-day moving averages. The change in implied volatility is defined in Equation (33). Log variance risk premium is defined by $LVRP_t \equiv \log(E_t^*(RV_{t+1})) - \log(E_t(RV_{t+1}))$, where $E_t(RV_{t+1})$ is estimated from the HAR-RV model

$$RV_{t+1} = \alpha + \beta_D RV_t + \beta_W RV_{t,week} + \beta_M RV_{t,mon} + \beta_V IV_t + \epsilon_{t+1}$$

and $E_t^*(RV_{t+1}) = IV_t/10000/252$. Open interest increase variable is equal to absolute change in open interest on the days when open interest increases and 0 otherwise. Open interest decrease variable equals the absolute change in open interest on the days when open interest decreases and 0 otherwise. Absolute return is the absolute value of futures return. Standard errors are given in parentheses. Coefficients significantly different from zero at 10 percent level, 5 percent level, and 1 percent level, are indicated by *, **, and ***, respectively. Both absolute return and variance risk premium are normalized by their sample means. The robust standard errors in parenthesis are calculated using Newey and West method. To save space, we omit the estimated coefficients for weekday dummies and the constant.

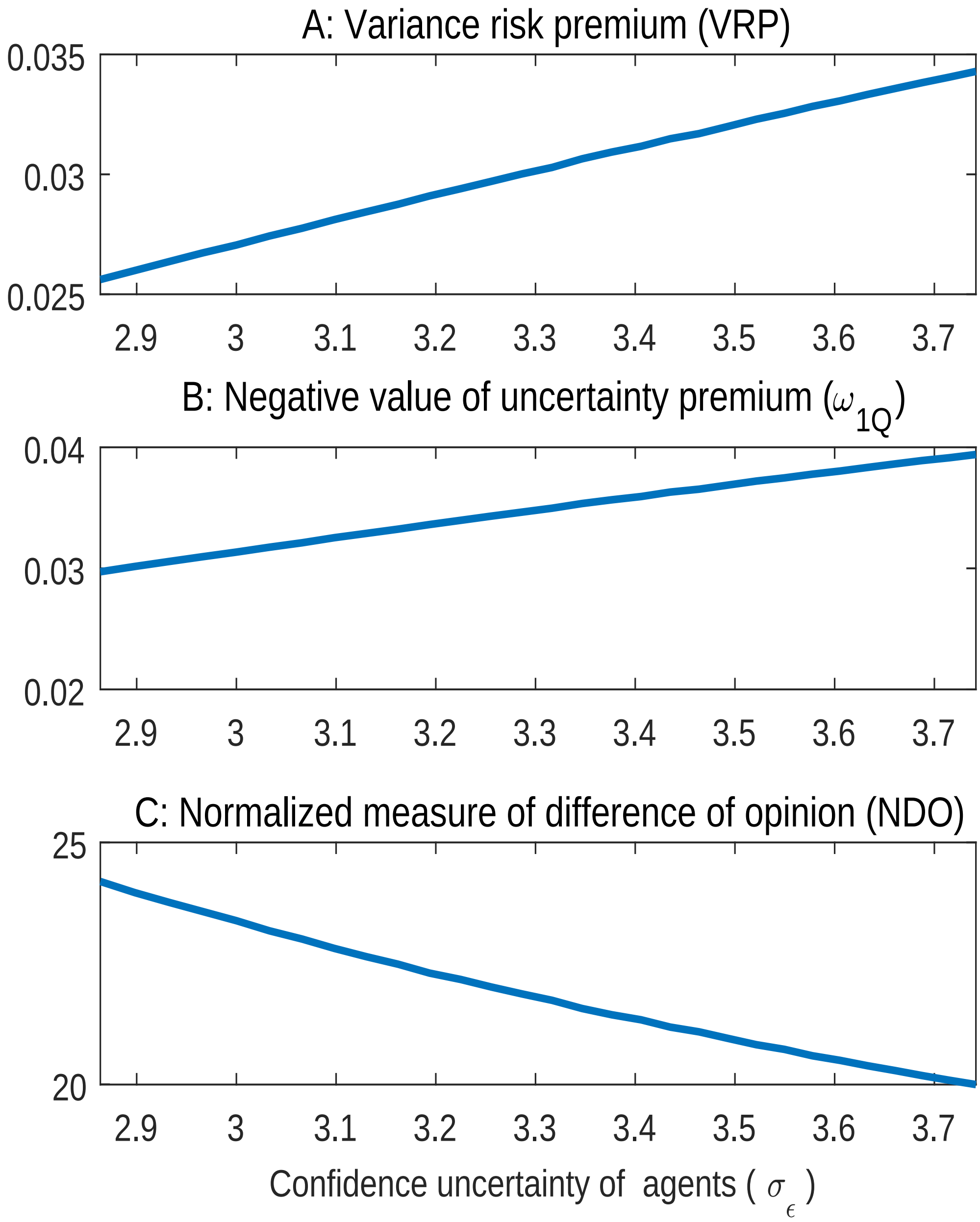

Figure 1: Effects of Agents' Confidence Uncertainty

Panels A, B, and C plot the variance risk premium VRP, the negative value of the uncertainty premium ω_{1Q}, and the normalized measure of difference of opinion NDO as functions of the uncertainty level of agents' confidence σ_ϵ. We set the risk tolerance coefficient $\tau = 1000$, the prior precision of the security payoff $K_1 = 11$, and the uncertainty level of the average agent's signal precision $\sigma_n = 1.86$.

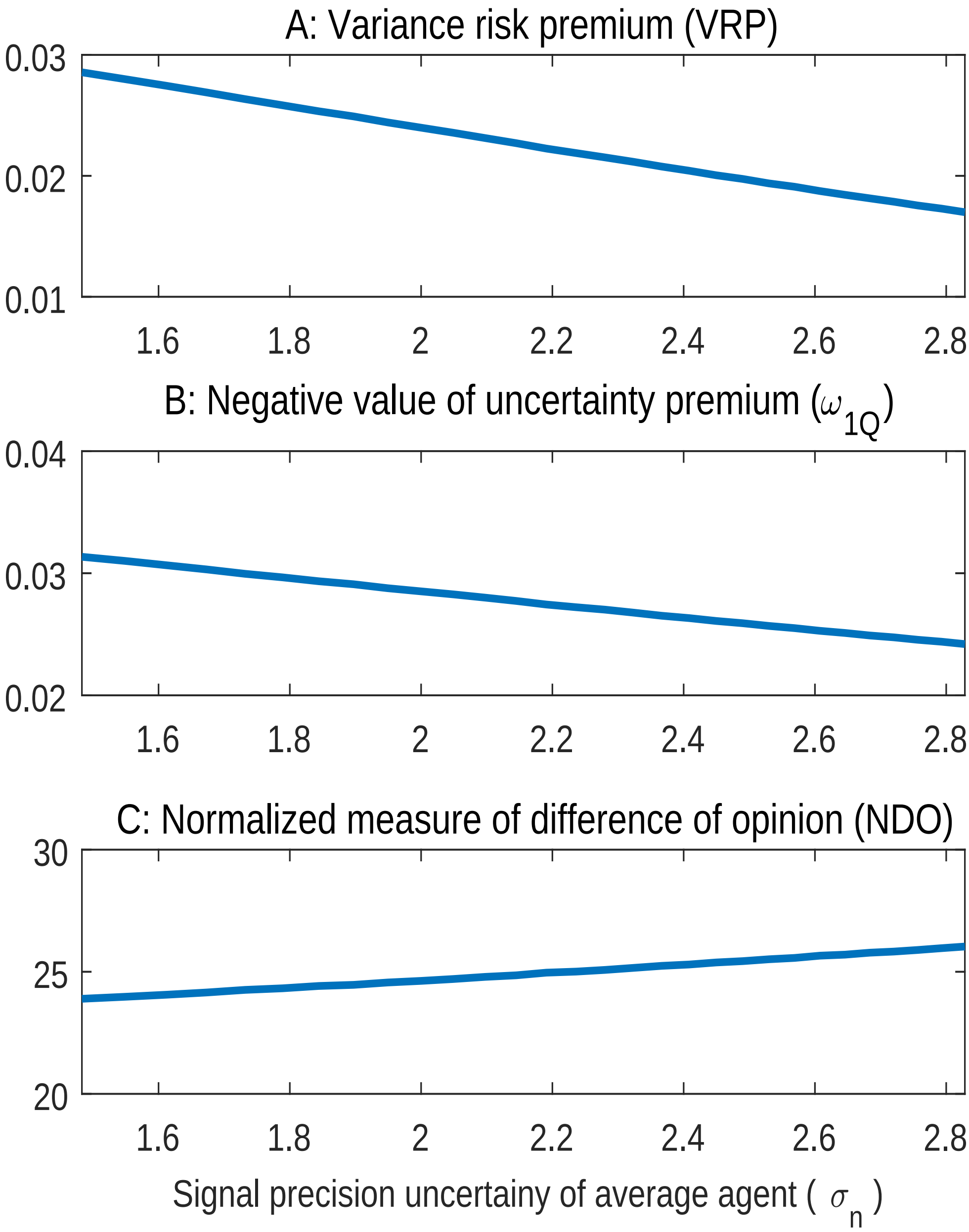

Figure 2: Effects of Uncertainty about Average Agent's Belief in Signal Precision

Panels A, B, and C plot the variance risk premium VRP, the negative value of the uncertainty premium ω_{1Q}, and the normalized measure of difference of opinion NDO as functions of the uncertainty level of the average agent's signal precision σ_n. We set the risk tolerance coefficient $\tau = 1000$, the prior precision of the security payoff $K_1 = 11$, and the uncertainty level of agent's confidence $\sigma_\epsilon = 2.83$.